An expression of stress field in 3D elastic medium using boundary integral equation method

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Abstract

We obtained off-plane stress expressions for a planar fault using boundary integral equation (BIE) in 3D infinite elastic medium, which are consistent with those by Tada (1997[9], 1998[10]). Assuming a 2D arbitrary shaped curved fault as a series of small planar fault patches, we discretised the kernels analytically for constant slip velocity within a patch. Taking a limit to the on-plane expression, our analytic kernels exactly agree with the solutions obtained by Fukuyama and Madariaga (1998)[5].

Introduction

In order to understand earthquake generation process, its background material structure as well as its shape become a significant factor for rupture propagation as pointed out for the 1992 Landers earthquake (Wald and Heaton, 1994[11]) and for the 1995 Kobe earthquake (Sekiguchi et al., 1996[8]). Thus to describe the precise rupture process on realistic non-planar faults in 3D elastic medium becomes very important. There have been many attempts for modeling earthquakes in 3D medium with finite difference method (FDM) (Olsen et al., 1997[7]). However, using FDM it is rather difficult to introduce precisely a constitutive relation on the fault, that essentially controls the rupture propagation (Matsu'ura et al, 1992[6]). In this paper, we use a boundary integral equation method (BIEM) which has advantages of using smaller number of elements to be solved and using more precise boundary condition on the fault which becomes important when introducing specific constitutive relation. Here we show the theoretical framework for numerical computation. In order to describe a non-planar fault in 3D medium, we may take two methodologies. The first one (Figure.1a) is to describe the fault model as rigorously as possible by introducing local coordinate system (Tada 1997 [9]). However, this will lead to a very complicated formulation in discretised form, which looks almost impossible to compute numerically. Another method that we used here is to approximate a curved fault plane with the combination of small planar patches as shown in Figure.1b. Since the stress on one of the planar patches (bold line) is attributed to the on-plane slip-velocity with on-plane kernel already derived (bold line) and to the off-plane slip-velocities with

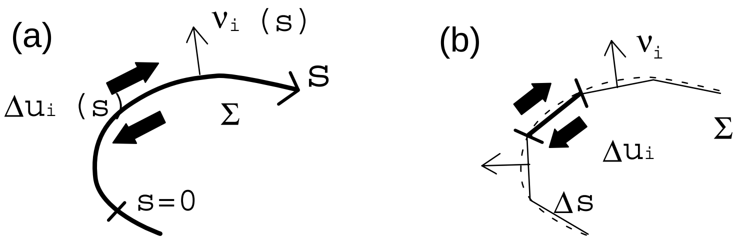

Figure 1: Two expressions for a curved fault Σ. (a) Describe a fault by introducing curved local axis (s) along the fault. (b) Approximate a curved fault with a series of small planar fault elements (Δs). In this case, off-plane stress expression for a flat source is required.

off-plane kernel (thin lines), we need to evaluate the off-plane kernel from a BIE for a planar fault exactly. First we derived the expressions of off-plane stress components and then obtained the analytical discrete kernel from them.

Formulation

It usually begins with a representation theorem, equation (3.2) in Aki and Richards (1980)[1], for example. The displacement field $u_i(\vec{x},t)$ in a (x_1,x_2,x_3)-coordinate system is written by a spatio-temporal convolution over the fault Σ

$$u_i(\vec{x},t) = \int_{-\infty}^{\infty} d\tau \int_\Sigma \Delta u_j(\vec{\xi},\tau) c_{jkpq} \nu_k(\vec{\xi}) \frac{\partial G_{ip}(\vec{x}-\vec{\xi},t-\tau)}{\partial \xi_q} d\Sigma \qquad (1)$$

where position $\vec{x}$ and time t are at an observation point in the medium, position $\vec{\xi}$ and time τ are on the source Σ. $\Delta\vec{u}(\vec{\xi},\tau)$ is slip discontinuity on Σ, $G(\vec{r},t-\tau)$ is a Green's function, $\vec{\nu}$ normal vector of Σ and c_{ijkl} elastic coefficient. Also we define hereafter that Latin and Greek subscripts vary from 1 to 3 and from 1 to 2, respectively, $r = \|\vec{r}\| \equiv \|\vec{x}-\vec{\xi}\|$, $\gamma_i \equiv (x_i - \xi_i)/r$, a comma between subscripts shows spatial derivatives $\partial/\partial x_i$ and overdots indicate time derivatives $\partial/\partial t$. We also use the summation convection rule. We suppose that a planar fault is embedded on $x_3 = 0$ plane in 3D homogeneous isotropic infinite elastic medium and slip discontinuity occurs only on the fault without tensile element ($\Delta u_3 \equiv 0$) and only after $t = 0$. The Green's function in equation (1) is called "Stokes tensor" given by equation (A1) in Appendix A.

For the stress representation obtained by spatial derivatives of equation (1);

$$\tau_{pq}(\vec{x},t) = -\int\int \Delta u_\zeta(\vec{\xi},\tau) c_{\zeta 3st} c_{pqin} \frac{\partial^2}{\partial x_n \partial x_t} G_{is}(\vec{r},t-\tau) d\Sigma d\tau, \qquad (2)$$

we use the similar regularisation method used by Cochard and Madariaga (1994)[3] on 2D anti-plane crack, Fukuyama and Madariaga (1995[4], 1998[5]) upon 3D on-planar crack for removing the very strong singularities due to second derivatives of Green's function in equation (2). Applying the techniques of Laplace transforms for time and partial integration, we get the regularised expression of stress systematically in the form such as a convolution of some derivatives of slip velocity and weak singular function. We show the complete expression for $\eta 3$-element in Appendix A.

Discretisation

We obtain the discretised kernels for equation (A3) in the case of a quadrantal dislocation on a plane fault;

$$\Delta \dot{u}_\eta = V_\eta^{lmn} H(\xi_1 - \xi_1^l) H(\xi_2 - \xi_2^m) H(\tau - \tau^n) \tag{3}$$

where $H(\cdot)$ is the Heaviside function, and then it enables us to consider a small plane patch, that is, the spatio-temporal grid $\xi_1^l - \Delta x/2 < \xi_1 < \xi_1^l + \Delta x/2$, $\xi_2^m - \Delta x/2 < \xi_2 < \xi_2^m + \Delta x/2$ and $\tau^n < \tau < \tau^n + \Delta t$ with uniform constant slip velocity V_η^{lmn}. We assume that space grid size Δx and time step Δt are uniform, and introduce a parameter κ as $\kappa = \Delta x / \Delta t$. Note that we define the observation point τ^{ijk} as $\tau(x_1^i, x_2^j, x_3; t^{k+1}) = \tau(i\Delta x, j\Delta x, h\Delta x; (k+1)\Delta t)$. We show the detailed form for the discretised kernel in Appendix B. Then finally we obtain a discretised kernel by using equation (6) of Fukuyama and Madariaga (1998)[5].

Now we model a system of not only a single flat fault but also several flat faults arranged arbitrarily which in some case approximates a curved fault. Steady numerical calculation requires that factor κ should be $\kappa \geq 2\alpha$ for avoiding seepage of the kernel to the next grid at the same time step, as already discussed by Fukuyama and Madariaga (1998)[5]. However, when we combined this method with slip- and time-dependent constitutive law on a plane, we could not help putting the factor $\kappa = 3\alpha$ (Aochi and Matsu'ura, 1998[2]). In the case of long time step computation, we didn't find the correct solution yet because the time resolution cannot chase the quick change of the constitutive law. We suspect that it depends mostly upon the nature of the law. Since we use a simple Newton-Raphson method as a root finding method, we should improve this or try other alternative methods.

Conclusion

We have derived the analytic expressions of off-plane stress in a discretised form which will be used in a simulation for a curved fault. Since most singularities have been removed by the regularisation, this kernel can stabilize our further numerical computation. For modeling a fully arbitrary shape in 3D, we will need to discretise them into triangular elements in the near future.

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Appendix A: Regularized Boundary Integral Equation

A Green's function in 3D homogeneous infinite elastic medium, Stokes tensor, is

$$\begin{aligned} G_{ij}(\vec{r},t) &= \frac{\beta^2}{4\pi\mu}\frac{3\gamma_i\gamma_j-\delta_{ij}}{r^3}\int_{r/\alpha}^{r/\beta} t'\delta(t-t')dt' \\ &+\frac{p^2}{4\pi\mu}\frac{\gamma_i\gamma_j}{r}\delta(t-\frac{r}{\alpha})-\frac{1}{4\pi\mu}\frac{\gamma_i\gamma_j-\delta_{ij}}{r}\delta(t-\frac{r}{\beta}) \end{aligned} \tag{A1}$$

where λ and μ are Lamé's constants, α and β are P- and S-wave velocities, respectively, $p=\beta/\alpha$ and $\delta(\cdot)$ is the Kronecker delta function. The Laplace form of the integration appeared in equation (A1);

$$\bar{I} = \int_0^\infty dt\, e^{-st} \frac{\beta^2}{r^2} \int_{r/\alpha}^{r/\beta} t'\delta(t-t')dt' = \frac{\beta^2}{r^2 s^2}\{(1+\frac{sr}{\alpha})e^{-sr/\alpha} - (1+\frac{sr}{\beta})e^{-sr/\beta}\} \tag{A2}$$

is useful for rewriting equation (2). As an example, we give the final expression of the $\eta 3$-component shear stress for planar shear crack;

$$\begin{aligned}
\tau_{\eta 3}(\vec{x},t) \quad = \quad & \frac{\mu}{4\pi}\int \frac{\gamma_\eta}{r^2}\Bigg[-12\frac{\beta^2}{r^2}\int_{r/\alpha}^{r/\beta} t'\Delta u_{\zeta,\zeta}(\vec{\xi},\|t-t'\|)dt' \\
& \quad - 4p^2\Delta u_{\zeta,\zeta}(\vec{\xi},\|t-\frac{r}{\alpha}\|) + 5\Delta u_{\zeta,\zeta}(\vec{\xi},\|t-\frac{r}{\beta}\|) \\
& \quad + \frac{r}{\beta}\Delta \dot{u}_{\zeta,\zeta}(\vec{\xi},\|t-\frac{r}{\beta}\|)\Bigg] d\Sigma \\
& - \frac{\mu}{4\pi\beta^2}\int \frac{1}{r}\Delta \ddot{u}_\eta(\vec{\xi},\|t-\frac{r}{\beta}\|)d\Sigma - \frac{\mu}{4\pi\beta}\int \frac{\gamma_\zeta}{r}\Delta \dot{u}_{\eta,\zeta}(\vec{\xi},\|t-\frac{r}{\beta}\|)d\Sigma \\
& - \frac{\mu}{4\pi}\int \frac{\gamma_\zeta}{r^2}\Delta u_{\eta,\zeta}(\vec{\xi},\|t-\frac{r}{\beta}\|)d\Sigma \\
& + \frac{\mu}{\pi}\int \frac{\gamma_3^2\gamma_\eta}{r^2}\Bigg[15\frac{\beta^2}{r^2}\int_{r/\alpha}^{r/\beta} t'\Delta u_{\zeta,\zeta}(\vec{\xi},\|t-t'\|)dt' \\
& \quad + 6p^2\Delta u_{\zeta,\zeta}(\vec{\xi},\|t-\frac{r}{\alpha}\|) - 6\Delta u_{\zeta,\zeta}(\vec{\xi},\|t-\frac{r}{\beta}\|) \\
& \quad + p^2\frac{r}{\alpha}\Delta \dot{u}_{\zeta,\zeta}(\vec{\xi},\|t-\frac{r}{\alpha}\|) - \frac{r}{\beta}\Delta \dot{u}_{\zeta,\zeta}(\vec{\xi},\|t-\frac{r}{\beta}\|)\Bigg] d\Sigma
\end{aligned} \tag{A3}$$

where $\|t-r/\alpha\|$ denotes that the slip functions Δu_i are evaluated only for positive time values of $(t-r/\alpha)$. In the limit $x_3 \to 0$, the spontaneous direct term $(\Delta \dot{u})$ is extracted from the 4th line and the off-plane terms among the 6th, 7th and 8th lines disappear.

Appendix B: Discretized Kernel for Semi-Infinite Slip Velocity

We summarized the discrete form of equation (A3) in the case of the source given by equation (3);

$$\begin{aligned}
\tau_{3\eta}^{ijk} \quad & = \tfrac{\mu}{4\pi\beta} \times \left[-V_\eta^{lmn}E^{ijklmn} + \right. \\
& + \textstyle\sum_{\zeta=1}^{2} V_\zeta^{lmn}\left\{P_{\eta\zeta}^{ijklmn}(\beta) + Q_{\eta\zeta}^{ijklmn}(\alpha) + R_{\eta\zeta}^{ijklmn}(\beta) - R_{\eta\zeta}^{ijklmn}(\alpha)\right\}\Big]
\end{aligned} \tag{B1}$$

Here E is contributed by the fourth and fifth lines in equation (A1), $P(\beta)$ and $Q(\alpha)$ by the first three lines related to S- and P-wave respectively, and $R(\beta)$ and $R(\alpha)$ by the last three lines. For the each kernels, we need to divide the solution into three cases. We introduce the following notations $\chi_1 = i-l$, $\chi_2 = j-m$, $\chi_3 = h$, $\Omega = k-n$, $\chi^2 = \chi_1^2+\chi_2^2+\chi_3^2$, ${\chi'_\eta}^2 = \chi_\eta^2 + \chi_3^2$ and $X_\eta(c) = \sqrt{(c/\kappa)^2\Omega^2 - {\chi'_\eta}^2}$, and define that $\bar{\eta} = 3 - \eta$, and c denotes either α or β. The three cases are the following;

$$\begin{aligned}
&\text{Case (I)} \;\; :(c/\kappa)^2\Omega^2 - {\chi'_\zeta}^2 > 0 \text{and} \chi_{\bar{\zeta}} < -X_\zeta(c) \quad , \text{ or } (c/\kappa)^2\Omega^2 - {\chi'_\zeta}^2 \leq 0 \\
&\text{Case (II)} \;:(c/\kappa)^2\Omega^2 - {\chi'_\zeta}^2 > 0 \text{ and } \chi^2 \leq (c/\kappa)^2\Omega^2 \\
&\text{Case (III)}:(c/\kappa)^2\Omega^2 - {\chi'_\zeta}^2 > 0 \text{ and } \chi_{\bar{\zeta}} > X_\zeta(c)
\end{aligned} \tag{B2}$$

and we gave the expression of E in equation (B1),

$$\begin{aligned}
E^{ijklmn} &= 2\pi H(\chi_1)H(\chi_2)H\left((\beta/\kappa)\Omega - |\chi_3|\right) \\
&+ \sum_{\zeta=1}^{2} \begin{cases} 0 & \text{for (I)} \\ I_\zeta(X_\zeta(\beta) + L_{\bar{\zeta}}(\beta)) - (\arctan \dfrac{X_\zeta(\beta)}{\chi_\zeta} + \arctan \dfrac{\chi_{\bar{\zeta}}}{\chi_\zeta}) & \text{for (II)} \\ 2I_\zeta X_\zeta(\beta) - 2\arctan \dfrac{X_\zeta(\beta)}{\chi_\zeta} & \text{for (III)} \end{cases}
\end{aligned} \tag{B3}$$

and the other terms in the case of $\eta = \zeta$;

$$P^{ijklmn}_{\eta\zeta(\eta=\zeta)}(\beta) = \begin{cases} 0 & ; \text{(I)} \\ -I_\eta \left[\dfrac{1}{3}X_\eta(\beta)(1 - 4K_\eta(\beta)) \right. & \\ \qquad \left. + L_{\bar{\eta}}(\beta)(1 - \dfrac{2}{3}(\beta/\kappa)^2\Omega^2 J_\eta)\right] & ; \text{(II)} \\ -\dfrac{2}{3}I_\eta X_\eta(\beta)(1 - 4K_\eta(\beta)) & ; \text{(III)} \end{cases} \tag{B4}$$

$$Q^{ijklmn}_{\eta\zeta(\eta=\zeta)}(\alpha) = \begin{cases} 0 & ; \text{(I)} \\ 2p^3 I_\eta \left[\dfrac{2}{3}X_\eta(\alpha)(1 - K_\eta(\alpha)) \right. & \\ \qquad \left. + L_{\bar{\eta}}(\alpha)(1 - \dfrac{1}{3}(\alpha/\kappa)^2\Omega^2 J_\eta)\right] & ; \text{(II)} \\ \dfrac{8}{3}p^3 I_\eta X_\eta(\alpha)(1 - K_\eta(\alpha)) & ; \text{(III)} \end{cases} \tag{B5}$$

$$R^{ijklmn}_{\eta\zeta(\eta=\zeta)}(c) = \begin{cases} 0 & ; \text{(I)} \\ 2\dfrac{\beta^3}{c^3}I_\eta \chi_3^2 \left[\dfrac{2}{3}\dfrac{X_\eta(c)}{{\chi'_\eta}^2}(1 - 4K_\eta(c)) \right. & \\ \qquad \left. + L_{\bar{\eta}}(c)\{J_\eta(1 - \dfrac{4}{3}K_\eta(c)) - \dfrac{(c/\kappa)^2\Omega^2}{\chi^4}\}\right] & ; \text{(II)} \\ \dfrac{8}{3}\dfrac{\beta^3}{c^3}I_\eta \chi_3^2 \dfrac{X_\eta(c)}{{\chi'_\eta}^2}(1 - 4K_\eta(c)) & ; \text{(III)} \end{cases} \tag{B6}$$

or in the case of $\eta \neq \zeta$;

$$P^{ijklmn}_{\eta\zeta(\eta\neq\zeta)}(\beta) = \begin{cases} 0 & ;\text{(I) and (III)} \\ -\frac{1}{3} + M(\beta)(1 - \frac{2}{3}M(\beta)^2) & ;\text{(II)} \end{cases} \tag{B7}$$

$$Q^{ijklmn}_{\eta\zeta(\eta\neq\zeta)}(\alpha) = \begin{cases} 0 & ;\text{(I) and (III)} \\ \frac{4}{3}p^3 - 2p^3 M(\alpha)(1 - \frac{1}{3}M(\alpha)^2) & ;\text{(II)} \end{cases} \tag{B8}$$

$$R^{ijklmn}_{\eta\zeta(\eta\neq\zeta)}(c) = \begin{cases} 0 & ;\text{(I) and (III)} \\ -2\frac{\beta^3}{c^3}\frac{\chi_3^2}{\chi^2}M(c)(1 - M(c)^2) & ;\text{(II)} \end{cases} \tag{B9}$$

where $I_\eta = \chi_\eta/{\chi'_\eta}^2$, $J_\eta = 1/\chi^2 + 2/{\chi'_\eta}^2$, $K_\eta(c) = (c/\kappa)^2\Omega^2/{\chi'_\eta}^2$, $L_\eta(c) = (c/\kappa)\Omega\chi_\eta/\chi$ and $M(c) = (c/\kappa)\Omega/\chi$.