Effect of Heterogeneity on Catastrophic Rupture

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Abstract

This paper reports a preliminary study on the rupture of a heterogeneous body loaded by its surrounding. It is known that stronger heterogeneity may prevent a body from rupture. However, when the surrounding is no longer rigid, the rupture shows a heterogeneity-surrounding coupled effect. In particular, the heterogeneity may induce inhomogeneous stress and damage, which may affect rupture occurrence.

Introduction

The importance of the heterogeneity in the crust is well known. Mogi pointed out that "successful prediction [of earthquakes] depends greatly on the heterogeneity of the area's structure" and "the possibility of earthquake prediction depends on whether or not precursory phenomena warning of a major fracture [large earthquake] occur and on whether or not it is possible to track down these phenomena" [1].

Hence, the mechanical formulation of heterogeneity and its effects are truly important in practice. In this paper, the heterogeneity of mesoscopic strength in an elastic-brittle medium is depicted by a distribution function. In terms of mean field approximation, we examined event-series related to rupture, such as damage localization, macroscopic maximum stress and eventual rupture of the medium loaded by its surrounding. Particularly, we found that rupture is closely dependent on the deviation of distribution function, surrounding as well as stress re-distribution in the concerned body.

Diversity of mesoscopic strength ------ Weibull distribution

We assume the spatial distribution of mesoscopic strength is statistically uniform. Then the main concern of heterogeneity will be (1) the diversity of meso-strength and (2) stress re-distribution (SRD) owing to the stress concentration and screening resulting from damage accumulation. Since mean field (MF) approximation is the easiest approach to the problem but has limitations, we shall apply both MF approximation and numerical simulation to examine these effects. An over-simplified but helpful road-map of the abovementioned factors and their effects with progress of deformation and damage is shown in Fig.1.

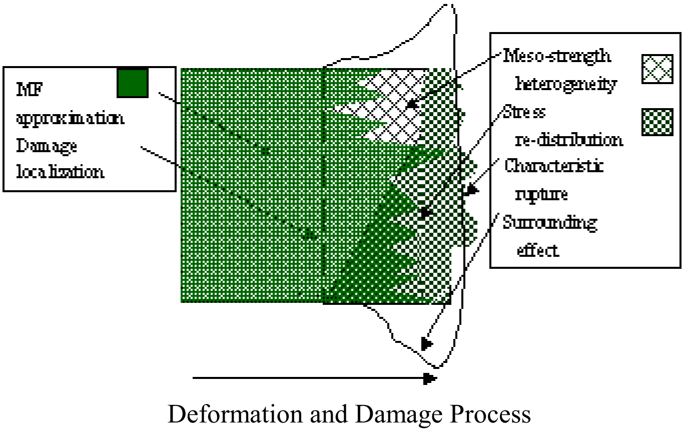

Figure 1: Road-map of effects of heterogeneity and surrounding, and the limitation of MF with progressive damage accumulation. MF is a helpful tool to deal with heterogeneity, surrounding and stress re-distribution, but with increasing damage the actual stress field may deviate from MF significantly and lead to specific rupture.

We examine such a medium consisting of mesoscopic elements possessing unique elastic modulus E_0 but diverse meso-strength. Suppose that all samples are macroscopically but statistically identical, though the samples have different mesoscopic configurations, as usually observed in laboratory. This implies that all these samples follow the same distribution function of meso-strength, $h(\sigma_c)$, where σ_c is the dimensionless strength normalized by its characteristic strength η, i.e. $\sigma_c = strength / \eta$. Similarly, normalized strain $\varepsilon_c = E_0 \cdot strain/\eta = \sigma_c$ and $h(\varepsilon_c) = h(\sigma_c)$. According to MF approximation, the macroscopic damage D of the heterogeneous sample can be expressed by

$$D = \int_0^{\sigma_s} h(\sigma_c) \cdot d\sigma_c = \int_0^{\frac{\sigma}{1-D}} h(\sigma_c) \cdot d\sigma_c \quad . \tag{1}$$

We assume that the true stress σ_s and the macroscopically nominal stress σ follow the relation $\sigma_s = \dfrac{\sigma}{1-D}$.

We adopt the Weibull distribution $w(\sigma_c; m)$ to be the distribution function, as usually done in practice, especially for rocks [2-3], where m is the shape factor (Weibull modulus),

$$w(\sigma_c) = m\sigma_c^{\,m-1} \exp[-(\sigma_c)^m] \quad . \tag{2}$$

Since the dimensionless and normalized stress and strain σ_c is equal to ε_c, we can also write the distribution function as $w(\varepsilon_c)$. The smaller the Weibull modulus m is, the more diverse the threshold is, which is to say the more heterogeneous the medium is. For instance, Weibull modulus m for rocks may range between 1- 5. Then the nominal stress-strain relation of such a medium can be easily obtained by means of MF approximation and is in reasonable agreement with experiment (Fig.2):

$$\sigma(\varepsilon) = \varepsilon(1 - D) = \varepsilon \cdot \int_{\varepsilon}^{\infty} h(\varepsilon_c) d\varepsilon_c = \varepsilon \cdot \exp(-\varepsilon^m) \quad . \tag{3}$$

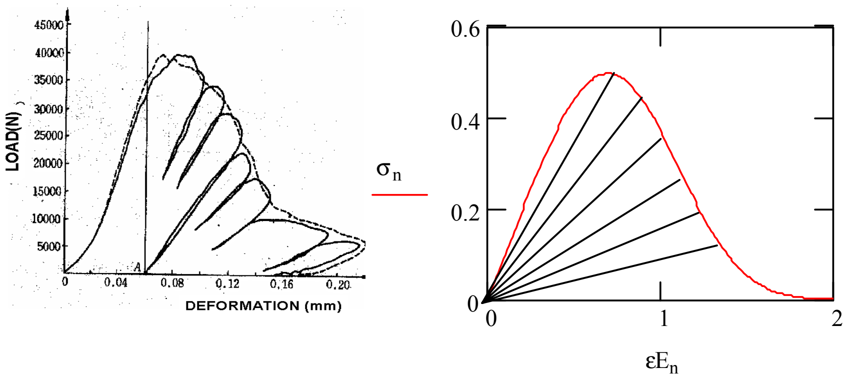

Figure 2: The stress and strain relation of the heterogeneous elastic-brittle model with Weibull modulus m = 3 (right). It appears to be a good approximate model of rock sample of Sandstone(left), (from Chinese Encyclopedia, Mechanics, p.529).

Event series – 1. Damage localization

Physically, damage localization implies the emergence of macroscopic inhomogeneity, i.e. a prelude to rupture. In previous work [4], we introduced the condition for damage localization as

$$\frac{\frac{\partial}{\partial T}\left(\frac{\partial D}{\partial X}\right)}{\frac{\partial D}{\partial X}} \geq \frac{\frac{\partial D}{\partial T}}{D} \quad . \tag{4}$$

For general function h and the Weibull distribution, the condition for damage localization is

$$D(1-D)^2\frac{h'}{h} \geq (1-D)(1-3D)h - \sigma h^2 \tag{5}$$

and

$$(m-1)D_L + m\log(1-D_L)[m(1-D_L)\log(1-D_L) + 1 - 2D_L] \geq 0 \tag{6}$$

respectively, where $h' = \frac{dh(\sigma_s)}{d\sigma_s}$.

Event series – 2. Macroscopic maximum stress

The slope of macroscopic stress-strain curve for general function h and the Weibull distribution is

$$\frac{d\sigma}{d\varepsilon} = [1 - \frac{\sigma h}{(1-D)^2}] \quad \text{and} \quad \frac{d\sigma}{d\varepsilon} = m\varepsilon^{m-1}e^{-\varepsilon^m} , \tag{7}$$

respectively. When the slope becomes zero, maximum stress is reached. This is also the critical point for rupture under load-control boundary (soft surrounding). For general function h and Weibull distribution, the conditions for maximum stress are

$$\varepsilon \cdot h = (1 - D) \quad \text{and} \quad \sigma_m = (em)^{-\frac{1}{m}} \quad \varepsilon_m = m^{-\frac{1}{m}} , \tag{8}$$

respectively.

The following Table lists corresponding values of damage D, stress σ and strain ε at damage localization (denoted by subscript L) and maximum stress (denoted by subscript m), respectively.

m	10	6	4	2
D_L	.009	.022	.046	.139
σ_L	.621	.520	.444	.333
σ_m	.719	.628	.551	.429
ε_L	.626	.532	.466	.387
ε_m	.794	.742	.707	.707

Event series – 3. Characteristic rupture

It is well known that in order to follow the whole process of deformation of rocks in laboratory it is necessary to apply a testing machine with high stiffness K_m [5]. This is because the elastic energy stored in the machine will be suddenly transferred into the rock specimen and induce its eventual rupture, as soon as the slope of load-displacement curve of the rock specimen becomes less than the negative of the machine stiffness $-K_m$:

$$\frac{dF}{du} \leq -K_m \quad . \tag{9}$$

Suppose $Km = k\ Ks_0$, where Ks_0 is the initial stiffness of specimen and is equal to ES/L_s, where E is Young's modulus, S is the sectional area and Ls is the length of the specimen, respectively. Then, the critical point for eventual rupture becomes

$$\frac{d\sigma}{d\varepsilon} \leq -k \quad , \tag{10}$$

where σ and ε are dimensionless stress and strain respectively, as above.

It appears that parameter k is the only additional parameter to govern the rupture. For example, if the slope of the stress – strain curve for a certain medium **never** becomes less than $-k$, i.e. its minimum

$$(\frac{d\sigma}{d\varepsilon})_{mim} > -k \quad , \tag{11}$$

there is no such a rupture at all. For the Weibull distribution, this leads to a minimum Weibull modulus $m_{c,}$

$$m_c \cdot \exp(-\frac{1+m_c}{m_c}) = k \quad , \tag{12}$$

which tells us that for $k = 1, m_c = 3.59$. This implies that if the sample has Weibull modulus less than 3.59, there will not be a rupture. This can explain the differences in observations made by Mogi (1985) concerning four materials: pine resin (homogeneous), Trachyte (nearly homogeneous), granite (heterogeneous) and pumice (extremely heterogeneous).

But one must keep in mind that the condition comes from two significant assumptions: k=1 and the mean field (MF) approximation. As a matter of fact, the parameter k includes the size effect and the MF approximation requires uniform stress distribution, which may be seriously violated when approaching rupture. So the catastrophic rupture is by no means so simple as the above two events: damage localization and maximum stress.

Effect of stress re-distribution(SRD) and numerical study

The obtained results are all based on mean field approximation. But, stress re-distribution may enhance deviation from MF differently under various loading conditions: tension, compression or shear. Unlike MF approximation, there is not only heterogeneous strength, but also stress concentration or screening owing to the spatial distribution of mesoscopic elements in numerical simulations. Now, let us have a look at some simulation results.

For two-dimensional simulations under tension, the whole process may be divided into three stages. In the initial stage, several small cracks occur and the stress-strain curve remains nearly linearly. Then with increasing damage, the high stress region may shift from one part to another very swiftly. This means that severe stress adjustment occurs and leads to immature macroscopic peak stress and rupture. In the final stage, the main crack forms, and high stress mainly adheres to the large crack tip. From the simulation, one can see that the stress re-distribution due to heterogeneity plays very significant role in rupture. However the unloading path seems to following the coupling with surrounding.

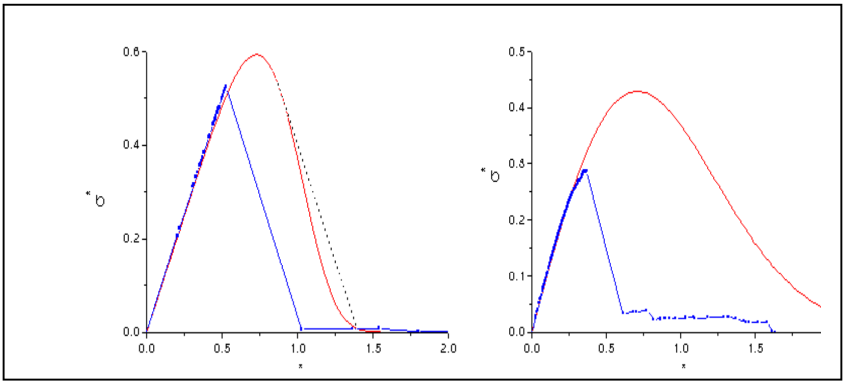

Figure 3: The curves of stress vs. strain for mean field model (higher one) and network simulations(lower one) for Weibull moduli m=5 (left) and m =2 (right), respectively.

In Fig.3, the curves of stress vs. strain for mean field model (higher one) and network simulation (lower one) are drawn for Weibull modulus m=5(left) and m=2(right) under tension. In the right diagram the result of mean field theory shows that there is no catastrophic rupture at all for m=2. But for network simulation the rupture really occurs. In the left picture, the dashed curve indicates the catastrophic rupture for mean field model, which is parallel to that from network simulation (the straight part), but the rupture occurred obtained from network simulation much earlier than that for the mean field model. From Fig.3, we can see that the deviations of stress from mean stress play an essential role in catastrophic rupture.

Size effect of surrounding on characteristic rupture

The stiffness *K*, elastic modulus *E* and the size *L* have the following relation, $K = \frac{ES}{L}$, where *S* and *L* are the sectional area and the length scale respectively. Suppose both specimen and surrounding have the same sectional area S. Then the condition for catastrophic rupture becomes

$$\frac{d\sigma}{d\varepsilon} \leq -\frac{E_m \cdot L_s}{E_{s_0} \cdot L_m}, \tag{13}$$

where *Ls* is the size of specimen and L_m is the size of surrounding, and E_m and Es_o are elastic moduli of the surrounding and specimen respectively.

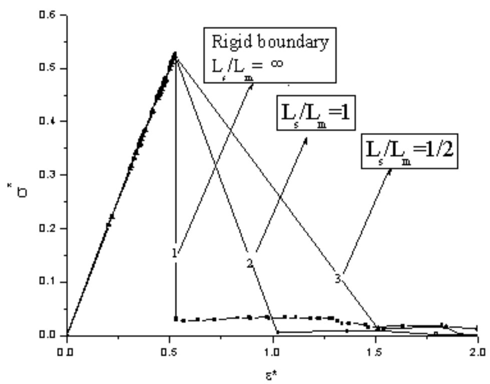

Fig.4 The curves of stress vs. strain for network simulation for ratio $\frac{L_S}{L_m} = \infty, 1, \frac{1}{2}$ respectively.

In Fig.4 The curves of stress vs. strain for network simulation for various ratios $\frac{L_S}{L_m}$ (i.e., for various stiffness ratio $\frac{K_S}{K_m}$) are drawn for Weibull modulus 5. The results show that the surrounding size effect is important

Conclusion

The effects of heterogeneity on catastrophic rupture may be a very complicated issue, closely coupled with surrounding, size and stress re-distribution. Generally speaking, (1) damage localization may depend on heterogeneity, which could be simply calculated based on the MF; and (2) catastrophic rupture, particularly macroscopically peak stress in simulation, demonstrates strong coupling effects of heterogeneity, surrounding and stress re-distribution.

Acknowledgments

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