Mechanics of Earth Crust with Fractal Structure

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Abstract

In the cases when the Earth's curst possesses self-similar structure it can be modelled by a sequence of continua each determined by the size of the averaging volume element. It is shown that tensorial properties and integral state variables scale by power laws with exponents common for all components of the tensors. Thus the scaling is always isotropic with anisotropy accounted for by the pre-factors. Real systems are not self-similar therefore the proposed approach is based on their approximation by self-similar systems. A necessary condition is formulated for such an approximation.

Introduction

Modelling geological phenomena using the concept of fractals has attracted considerable interest (eg, [1]-[3]) because it offers a rational method of dealing with such a highly irregular structure as the Earth's crust. The method is based on developing power scaling laws pertinent to different aspects of the Earth's crust behaviour. When the mechanics of fractal objects is considered, two essential problems arise. First, the highly irregular objects no longer permit the introduction of a scale at which they can be treated using the methods of continuum mechanics. Even the basic notions, such as stress and strain cannot be introduced. For instance, the stress renormalisation accounting for the fractal scaling of elementary area [4] leads to non-traditional units of stress that can even depend on the position within the material. On the other hand, the methods of discontinuous mechanics (like the distinct element method) become computationally prohibitive when multi-scale objects are modelled. Thus, a rational method is needed that can reconcile the irregular nature of fractal objects with the highly developed machinery of continuum mechanics. Second, the fractal description is based on the idea that the object is self-similar. This is a very strong property leading to the conclusion that all functions which are functions of the scale must only be power functions. In reality however, the natural dependencies are only approximated by the power functions. Then the conditions have to be formulated in which the power-law approximation is consistent with the notion of self-similarity.

The present paper aims at addressing these two problems.

Continuum multi-scale modelling of self-similar objects

The main obstacle for modelling a fractal material by the tools of continuum mechanics is the absence of a characteristic scale which means that the material possesses discontinuities of all scales. In order to overcome this obstacle consider a material, say a volume of the Earth's crust with fractal structure, chose a scale, H, and remove all discontinuities of the size H and greater. Thus we obtain a material with truncated structure as well as a scale at which the truncated material can be modelled as a continuum. Therefore, the actual stress computations can be conducted for an object that is not fractal but rather a continuum that models the material with certain structure of discontinuities of the sizes smaller than H. By varying H one can model the self-similar fractal objects by a set of continua each of them being characterised by its own yardstick, H, specifying the scale of the continuum. Each continuum models the fractal material at the scale H in the sense that the volume elements of size H cut from both the original material and the modelling continuum respond equally to uniform loading. The yardstick H determines the resolution: no features with all dimensions smaller than H are viewable in the H-continuum, Dyskin, (2001)[5]. Thus the H-continuum replaces the original material with the one possessing modified microstructure in which only those microstructural elements are present that have characteristics sizes less than H.

Since the fractal objects have no characteristic length, a continuum set of scales, $H>0$, should be used. Thus self-similar fractal objects should be modelled by a continuum of continua all characteristics being functions of H. If these functions do not change sign then, according to the general theorem (eg, [1]) they must be the power functions, $f(H)=f^*H^\alpha$, where f^* is a prefactor.

$$f_{ijk\ldots}(H) = f^*{}_{ijk\ldots} H^{\alpha_{ijk\ldots}}, \alpha_{ijk\ldots} = \alpha = const \tag{1}$$

This is held because (a) the tensorial property implies that by coordinate rotations all tensorial components should be linearly transformed and (b) the power functions with different exponents are linearly independent.

In particular, if the modelling continua are linearly elastic, the case considered hereafter, the tensors of general anisotropic moduli, C_{ijkl}, and compliances, A_{ijkl}, should scale as

$$C_{ijkl}(H) = c_{ijkl} H^{\alpha}, A_{ijkl}(H) = \alpha_{ijkl} H^{\beta}, i,j,k,l = 1,2,3 \tag{2}$$

Therefore, the tensors of elastic moduli and compliances must scale isotropically. This property is independent of the microstructure meaning that no matter what the anisotropy of the material is the scaling must be isotropic. The anisotropy is accounted for by the prefactors, c_{ijkl} and a_{ijkl}. The particular values of α, β, c_{ijkl} and a_{ijkl} depend on the material structure. The case of parts of the Earth's crust with self-similar crack distributions is considered in the following section.

Scaling laws for elastic moduli in Earth's crust with self-similar distributions of cracks

Consider a material containing a self-similar distribution of cracks:

$$f(R) = wR^{-4}\int_{R_{\min}}^{R_{\max}} R^3 f(R)dR = v_t \tag{3}$$

where, w is the concentration factor chosen to ensure the specified total concentration, v_t. This distribution has a remarkable property no other self-similar distribution possesses that will make it possible to suggest an accurate procedure of computing the effective moduli as power functions of the crack size. Consider the probability, $P(n)$, that in a vicinity of an crack of size R, ie a region of size proportional to R, there are cracks of smaller sizes, say from R/n to R, where $n>1$: $P(n)$~w(n^3-1), ie it does not depend on the crack size, R. Since (3) represents real distributions only asymptotically as $R_{max}/R_{min}\to\infty$, ie as $w\to 0$ (v_t=const), for any n the value of w can be chosen small enough to make the probability negligible for any crack size.

This property can be interpreted in the sense that for any crack the probability to find nearby a crack of a similar size is negligible; only cracks of greatly different sizes can be found there. Mechanically it means that the interaction between the cracks of similar sizes can asymptotically be neglected; only interaction between cracks of very different sizes is to be taken into account. This justifies the use of property of the differential self-consistent method for calculating the effective characteristics considered by Salganik (1973)[6].

According to the method, the compliance increment ΔA_{ij}, i,j=1, …, 6, at each scale is determined by the contribution of non-interacting cracks considered in an effective continuum. This contribution is proportional to the concentration of the group of cracks at hand, wdH/H. Therefore

$$A_{ij}(H+dH) = A_{ij}(H) + wS_{ij}(A_{11},...,A_{66})dH/H \tag{4}$$

Here S_{ij} is a homogeneous function of the first degree specific for the given geometry and distribution of parameters of the cracks.

A similar equation can be written in terms of elastic moduli. Substituting (2) into (4) or its analogue for elastic moduli one obtains the following scaling equations

$$\beta a_{ij} = wS_{ij}(a_{11},a_{12}...,a_{66}), \alpha c_{ij} = w\Lambda_{ij}(c_{11},c_{12}...,c_{66}), i,j=1...6 \tag{5}$$

where function Λ_{ij} represents the contribution of cracks to the elastic moduli at each step of the self-consistent method. These equations will be solved for two specific cases.

Isotropically oriented disk-like cracks

In the case of randomly oriented disk-like cracks, the material is isotropic. Then, the components of function Λ_{ij} can be extracted from the expressions for effective Young's modulus, E, and Poisson's ratio, ν, for non-interacting cracks (eg, [6]):

$$A = E_m\left[1-\frac{16}{45}(10-3v_m)\frac{1-v_m^2}{2-v_m}v\right], v = v_m\left[1-\frac{16}{15}(3-v_m)\frac{1-v_m^2}{2-v_m}v\right] \tag{6}$$

where E_m and ν_m are the Young's modulus and Poisson's ratio of the material. The solution of the second equation in (5) gives the following scaling law

$$\nu = 0, \quad E = eH^{\alpha}, \quad \alpha = -16w/9 \tag{7}$$

Plane with two mutually orthogonal sets of cracks

Consider now a 2-D problem for a plane with two mutually orthogonal sets of cracks. The self-similar distribution function in 2-D has the form

$$f(l) = \omega l^{-3} \tag{8}$$

where, w is the concentration factor ensuring the specified total concentration, l is the crack length. It is assumed that the set of cracks perpendicular to the x_i axis is characterised by the distribution ω_i/l^3 such that the total distribution is ω/l^3 with the concentration factor $\omega = \omega_1 + \omega_2$.

We consider Vavakin and Salganik's (1978) [7] solution for the effective compliances for an orthotropic plate with a set of non-interacting cracks aligned to one of the material's symmetry axes and generalise it to two sets of non-interacting cracks and using the method outlined above one obtains the scaling equations (see Dyskin 2002 [8] for details):

$$\begin{cases} \beta a_{ii} = \dfrac{\pi}{4}\omega_i \sqrt{a_{ii}(a_{66} + 2\sqrt{a_{11}a_{22}})}, \quad i = 1,2 \\ \beta a_{66} = \dfrac{\pi}{4}\omega_2 \sqrt{a_{11}(a_{66} + 2\sqrt{a_{11}a_{22}})} + \dfrac{\pi}{4}\omega_1 \sqrt{a_{22}(a_{66} + 2\sqrt{a_{11}a_{22}})} \end{cases} \tag{9}$$

These equations have a unique solution that produces the following scaling laws:

$$\begin{aligned} &A_{ij} = a_{ij}H^{\beta}, \quad \beta = (\pi/2)\sqrt{\omega_1\omega_2} \\ &a_{22} = a_{11}(\omega_2/\omega_1)^2, \quad a_{66} = 2a_{11}(\omega_2/\omega_1), \quad a_{12} = 0 \end{aligned} \tag{10}$$

As the concentration of one of the sets vanishes, say $\omega_2 \to 0$, the exponent and all compliances except a_{11} vanish. The material becomes completely rigid in direction x_2.

Self-similar approximation

The fact that all components of the tensorial functions of H must scale with the same exponent suggests that the property of self-similarity is very restrictive reaching beyond the power law. This implies that not all systems which show dependencies close to the power ones can be approximated by self-similar systems. In order to derive conditions under which such an approximation is possible, we first analyse the way the fractal concept is applied to real systems.

Suppose a system has a property which scales as $f(H)>0$. This function is then plotted in log-log coordinates and, if the plot has a region close to a straight line, the fractal property is pronounced to be held within this region. Mathematically this corresponds to representing the function as $\xi = \ln f(\exp(\eta))$. This is a function of η. If it is sufficiently smooth, by linearising it in a vicinity of $\eta_o = \ln H_o$ and returning to the original variables one obtains the self-similar approximation:

$$f(H) \approx f(H_0)\left(\frac{H}{H_0}\right)^{\alpha}, \alpha = \frac{f'(H_0)H_0}{f(H_0)}, f(H_0) \neq 0 \tag{11}$$

Thus one can always approximate any function by a power one at list in a sufficiently narrow vicinity of H_o. Obviously, the formula generalises to the case $f(H)<0$.

Real systems shall of course be characterised by a set of scale-dependent properties. If there is a subset of properties with the same units $f_1(H)$, $f_2(H)$, … then by summation multiplication and multiplication by a dimensionless number new properties can be derived (eg rotation of the tensor of elastic moduli). Such properties that permit summation and multiplication will be here called *compatibl*e. (All other functions on the set of compatible properties can be reduced to may be infinite sequences of summations and multiplications.) Therefore, the self-similar approximation to be meaningful should preserve these operations. This means that the self similar approximation of the product and the sum of two functions in a vicinity of the same point, H_0, should be the product and the sum of the corresponding approximations respectively. We shall check this property now. (The case of multiplication by a dimensionless number is trivial.) Suppose

$$f_i(H) \approx f_i(H_0)\left(\frac{H}{H_0}\right)^{\alpha_i}, \alpha = \frac{f_i'(H_0)H_0}{f_i(H_0)} \tag{12}$$

Then for multiplication we have the necessary property:

$$f_1(H)f_2(H) \approx f_1(H_0)f_2(H_0)\left(\frac{H}{H_0}\right)^{\alpha_1+\alpha_2} \tag{13}$$

However for summation a similar property

$$f_1(H) + f_2(H) \approx [f_1(H_0) + f_2(H_0)]\left(\frac{H}{H_0}\right)^{\alpha} \tag{14}$$

only holds when

$$\alpha_1 = \alpha_2 \quad \text{or} \quad \frac{f_1'(H_0)}{f_1(H_0)} = \frac{f_2'(H_0)}{f_2(H_0)} \tag{15}$$

Equation (15) establishes equivalence between the properties. Thus only systems that are described by sets of *compatible* properties equivalent in the sense of (15) can be approximated by self-similar ones. This is a *necessary* condition of self-similar modelling.

Conclusions

The proposed multi-scale modelling of mechanical behaviour of the Earth's crust with self-similar structure is based on representing it as a continuum of continua of different scales. In such a modelling scaling of mechanical properties i.e. transition from one continuum to another is described by power laws. The tensorial mechanical properties scale by power laws with exponents common for all components of the tensors.

Real systems are not self-similar, so one can only talk about their approximation by the self-similar ones. Such an approximation is based on representing all scale-dependent properties in log-log coordinates and linearising them in a vicinity of a certain scale. This provides a local (in terms of scale) power law approximation. When the real system is described by a set of properties, each of them should be approximated in a vicinity of the same scale. However, the self-similar approximation is only possible when all compatible properties (the ones which permit summation and multiplication) have power law approximation with the same exponent.

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