Finite-Element Simulation of Seismic Wave Propagation with a Voxel Grid

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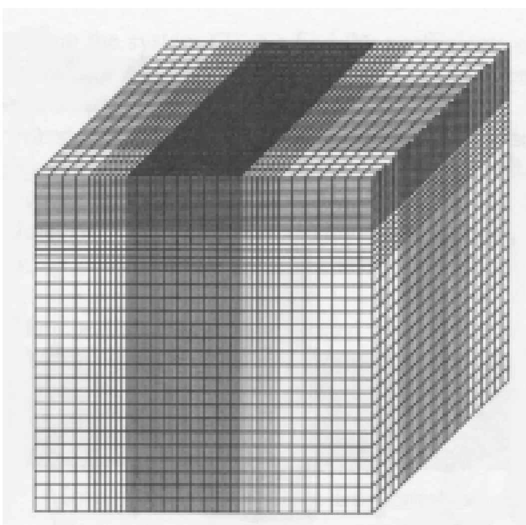
Finite Difference VS Finite Element

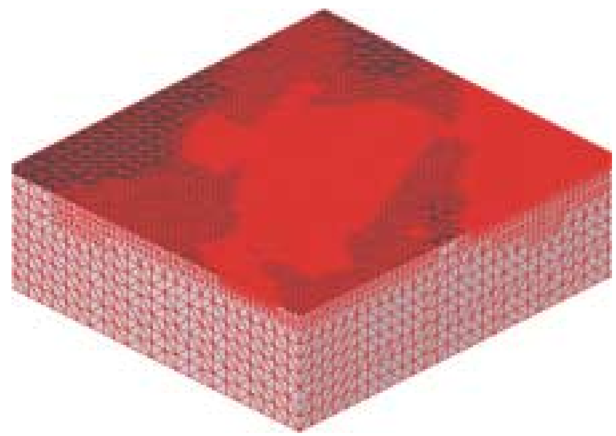
Comparison for seismic ground motion simulation

	Memory	Relative CPU Time	Discretization	Stiffness	Free Surface
FDM	20MB / Mdof	1	Discrete Grid	isotropic	approximate
FEM*	400MB / Mdof	Several Tens	Tetrahedra *etc*	anisotropic	by nature

* estimated from Bao et al. (1998) dof = degree of freedom

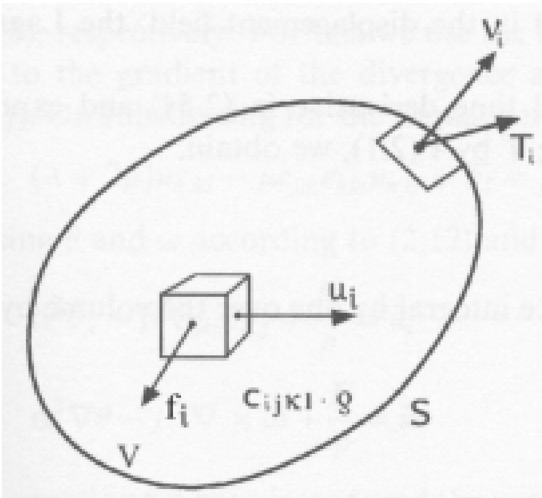


Finite Difference
(from Pitarka, 1999)



Finite Element
(from Bao *et al*., 1998)

Theoretical Foundation



$$V = \sum_n v_n$$

Principle of Virtual Work (for any δu_i)

$$\int \sigma_{ij} \delta\varepsilon_{ij} dV = \int f_i \delta u_i dV + \int T_i \delta u_i dS$$

Finite Elements v_n and shape functions N_k

$$u_i(x,y,z) = \sum_k N_k(x,y,z) u_{ik}$$

by the Galerkin method, if $T_i = 0$

The solution of Eq (1) satisfies $T_i = 0$ by nature.

$$\mathbf{M}\frac{d^2\mathbf{u}}{dt^2} + \mathbf{C}\frac{d\mathbf{u}}{dt} + \mathbf{K}\mathbf{u} = \mathbf{f} \quad (1)$$

nodal displacement vector $\mathbf{u} = \{u_{ik}\}$

nodal force vector $\mathbf{f} = \{f_{ik}\}$

mass matrix $\mathbf{M} = \{m_{kl}\}, \quad m_{kl} = \int N_k \rho N_l dv_n$

Explicit Method

$$\mathbf{M}\frac{d^2\mathbf{u}}{dt^2}+\mathbf{C}\frac{d\mathbf{u}}{dt}+\mathbf{K}\mathbf{u}=\mathbf{f}$$

Common

- Mass lumped at the nodes $\quad \mathbf{M}=\mathrm{diag}\{m_{kk}\}$
- 2nd-order central difference $\quad \frac{d^2\mathbf{u}}{dt^2}\approx\frac{\mathbf{u}_{t+\Delta t}-2\mathbf{u}_t+\mathbf{u}_{t-\Delta t}}{\Delta t^2}$

Option 1

- $\mathbf{C}\mathbf{u}_{t+\Delta t}\approx\mathbf{C}_1\mathbf{u}_{t+\Delta t}+\mathbf{C}_2\mathbf{u}_t, \quad \mathbf{C}_1=\mathrm{diag}\{c_{kk}\}, \mathbf{C}_2=\mathbf{C}-\mathbf{C}_1$
- 1st-order central difference $\quad \frac{d\mathbf{u}}{dt}\approx\frac{\mathbf{u}_{t+\Delta t}-\mathbf{u}_{t-\Delta t}}{2\Delta t}$

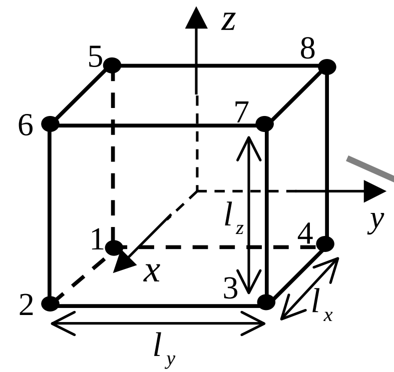
$$\mathbf{u}_{t+\Delta t}=(\mathbf{M}+\Delta t\mathbf{C}_1/2)^{-1}\left\{\Delta t^2\mathbf{f}+\left(2\mathbf{M}-\Delta t\mathbf{C}_2/2-\Delta t^2\mathbf{K}\right)\mathbf{u}_t-(\mathbf{M}-\Delta t\mathbf{C}/2)\mathbf{u}_{t-\Delta t}\right\}$$

Option 2

- full damping matrix
- 1st-order backward difference $\quad \frac{d\mathbf{u}}{dt}\approx\frac{\mathbf{u}_t-\mathbf{u}_{t-\Delta t}}{\Delta t}$

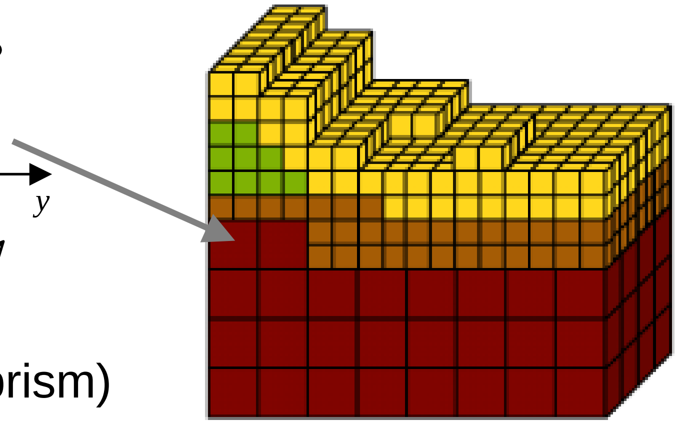
$$\mathbf{u}_{t+\Delta t}=\mathbf{M}^{-1}\left\{\Delta t^2\mathbf{f}+\left(2\mathbf{M}-\Delta t\mathbf{C}-\Delta t^2\mathbf{K}\right)\mathbf{u}_t-(\mathbf{M}-\Delta t\mathbf{C})\mathbf{u}_{t-\Delta t}\right\}$$

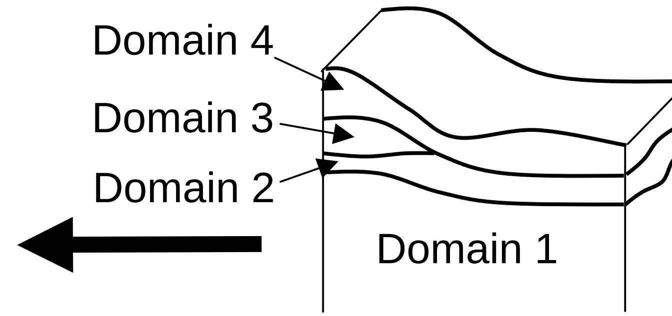
Voxel Elements & Domain Partitioning



Voxel (rectangular prism)

$1, 2, 3, \cdots =$ nodes





Linear Shape Functions

$$N_1 = \frac{1}{8}\left(1-\frac{2x}{l_x}\right)\left(1-\frac{2y}{l_y}\right)\left(1-\frac{2z}{l_z}\right)$$

$$N_2 = \frac{1}{8}\left(1+\frac{2x}{l_x}\right)\left(1-\frac{2y}{l_y}\right)\left(1-\frac{2z}{l_z}\right)$$

$$\vdots$$

$$N_8 = \frac{1}{8}\left(1-\frac{2x}{l_x}\right)\left(1+\frac{2y}{l_y}\right)\left(1+\frac{2z}{l_z}\right)$$

Mesh Generation

As easy as that for the finite difference method.

Domain Partitioning

Only a domain ID, not material parameters or node coordinates, is memorized for each element to reduce memory requirement.

Stiffness & Damping

Hooke's Law

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon} + \boldsymbol{\eta}\frac{d\boldsymbol{\varepsilon}}{dt}$$

→

stiffness matrix $\mathbf{K} = \int \mathbf{B}^{\mathrm{T}}\mathbf{D}\mathbf{B}\,dv_n$

damping matrix $\mathbf{C} = \int \mathbf{B}^{\mathrm{T}}\boldsymbol{\eta}\mathbf{B}\,dv_n$

$$\mathbf{B} = \{\partial N_k / \partial x_i\}$$

If the medium is isotropic

$$\mathbf{D} = \begin{bmatrix} D_{11} & D_{12} & D_{12} & 0 & 0 & 0 \\ D_{12} & D_{11} & D_{12} & 0 & 0 & 0 \\ D_{12} & D_{12} & D_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & D_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & D_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{44} \end{bmatrix} \quad \boldsymbol{\eta} = \begin{bmatrix} \eta_{11} & \eta_{12} & \eta_{12} & 0 & 0 & 0 \\ \eta_{12} & \eta_{11} & \eta_{12} & 0 & 0 & 0 \\ \eta_{12} & \eta_{12} & \eta_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & \eta_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \eta_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \eta_{44} \end{bmatrix}$$

$$D_{11} = V_P^2\rho, \quad D_{44} = 2V_S^2\rho, \quad D_{12} = D_{11} - D_{44}$$

$$\eta_{11} = \frac{D_{11}}{2\pi f Q_P} \qquad \eta_{44} = \frac{D_{44}}{2\pi f Q_S} \qquad \eta_{12} = \eta_{11} - 2\eta_{44}$$

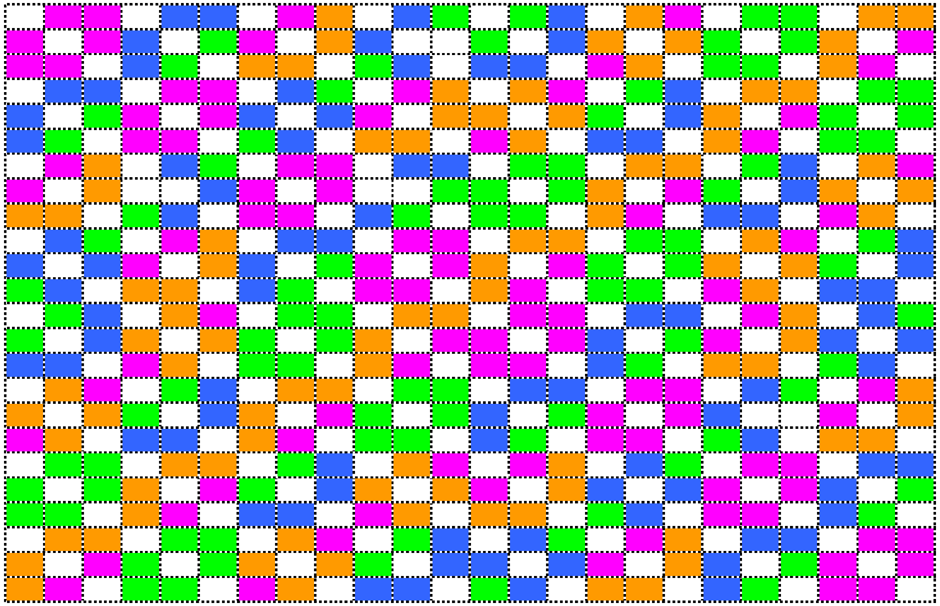
f : reference frequency

(Auld, 1990)

Isotropic Voxels

$$\mathbf{K} = \int \mathbf{B}^{\mathrm{T}} \mathbf{D} \mathbf{B} \, dv_n$$

The stiffness matrix has 24 independent components for full anisotropy.



According to the symmetry in the isotropic elastic constants and linear shape functions, K has the above symmetry.
The number of independent components is reduced to 12.

Finite Difference VS Finite Element (2)

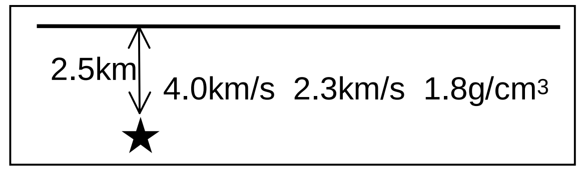
Comparison for seismic ground motion simulation

	Memory	Relative CPU Time	Discretization	Free Surface	Stiffness
FDM	20MB / Mdof	1	Discrete Grid	Approximate	isotropic
Voxel FEM	20MB / Mdof	2.8	Voxel	by nature	anisotropic
Voxel FEM	20MB / Mdof	1.4	Voxel	by nature	isotropic

Conclusions of the Formulation

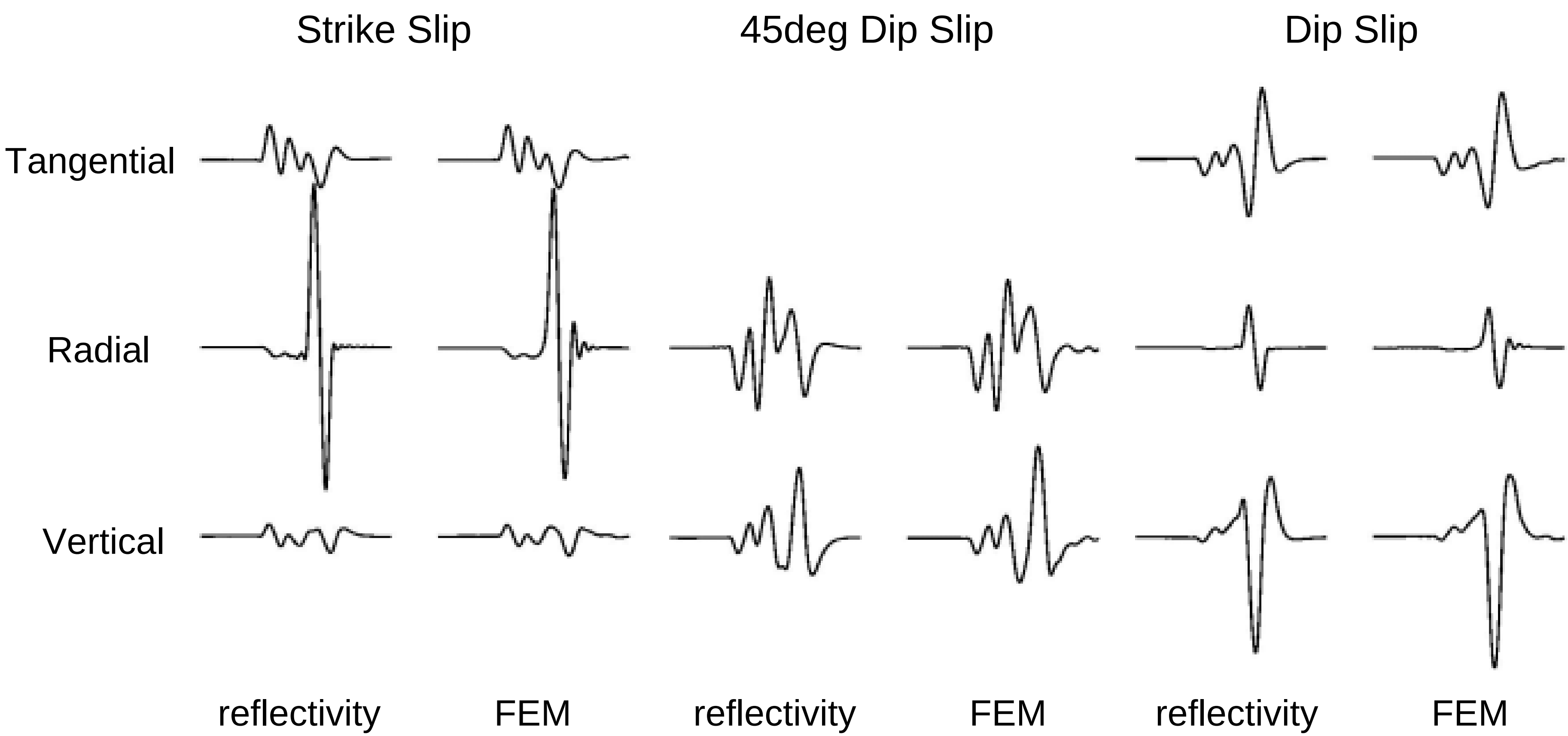
- The voxel grid and domain partitioning make FEM several times smaller and faster than usual.
- The explicit isotropy doubles the speed of FEM.
- Restricting the elements to voxels results in losing the flexibility of modeling in some degree. Discretization error will be introduced if a very complex structure is approximated with voxels.

Verification (halfspace)

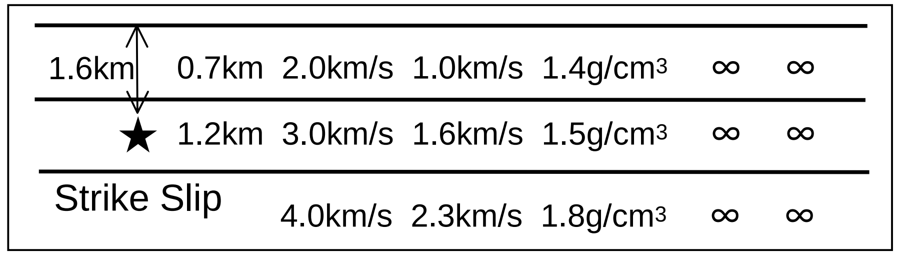


30 x 30 x 10km volume
4,608,000 elements (125m interval)
20s 800 step (0.025s interval)

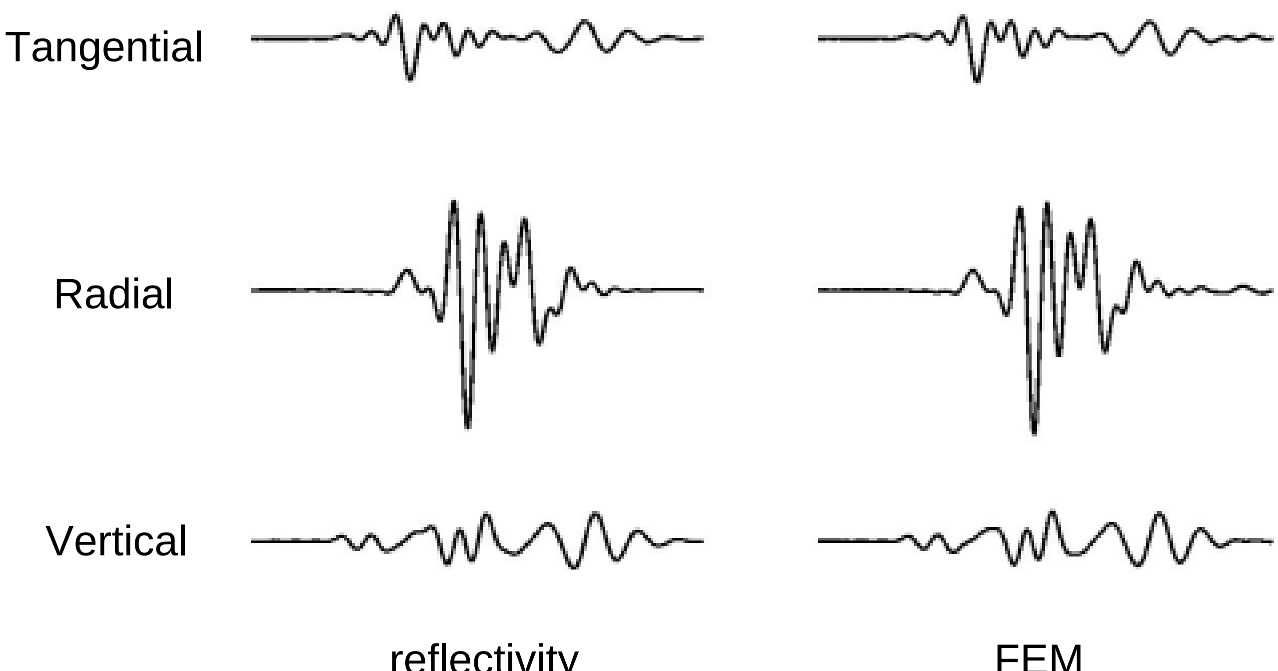
1.4 hour, 243MB on Windows 2000
by a Pentium 4 (1.7GHz), 1GB RIMM



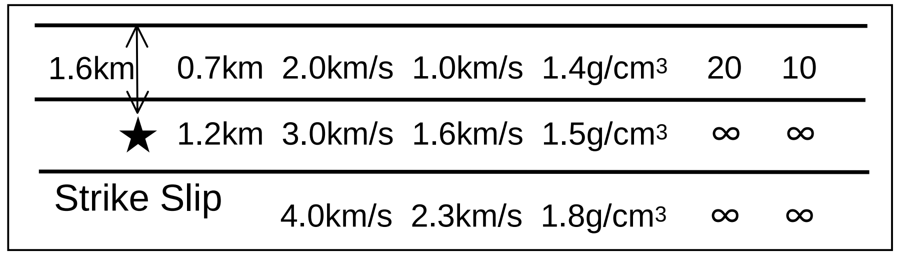
Verification (layered)

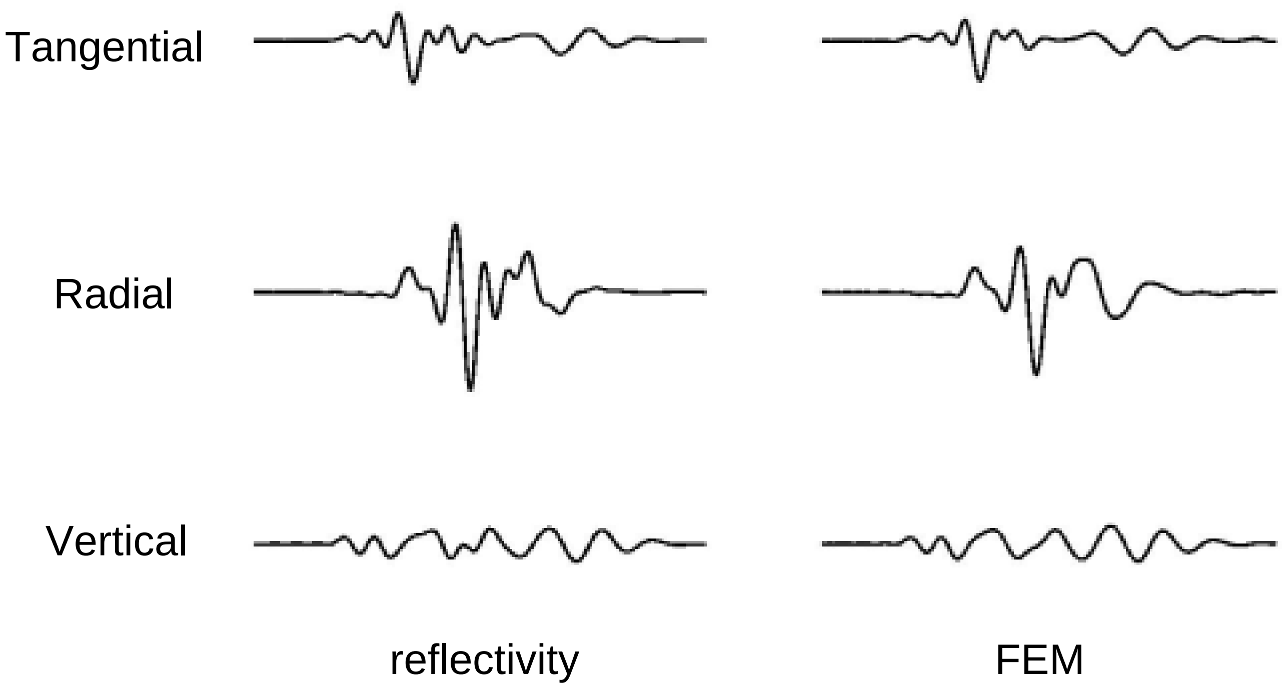


675MB memory for 12,800,000 elements

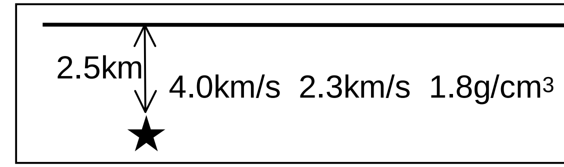


Verification (layered+Q)



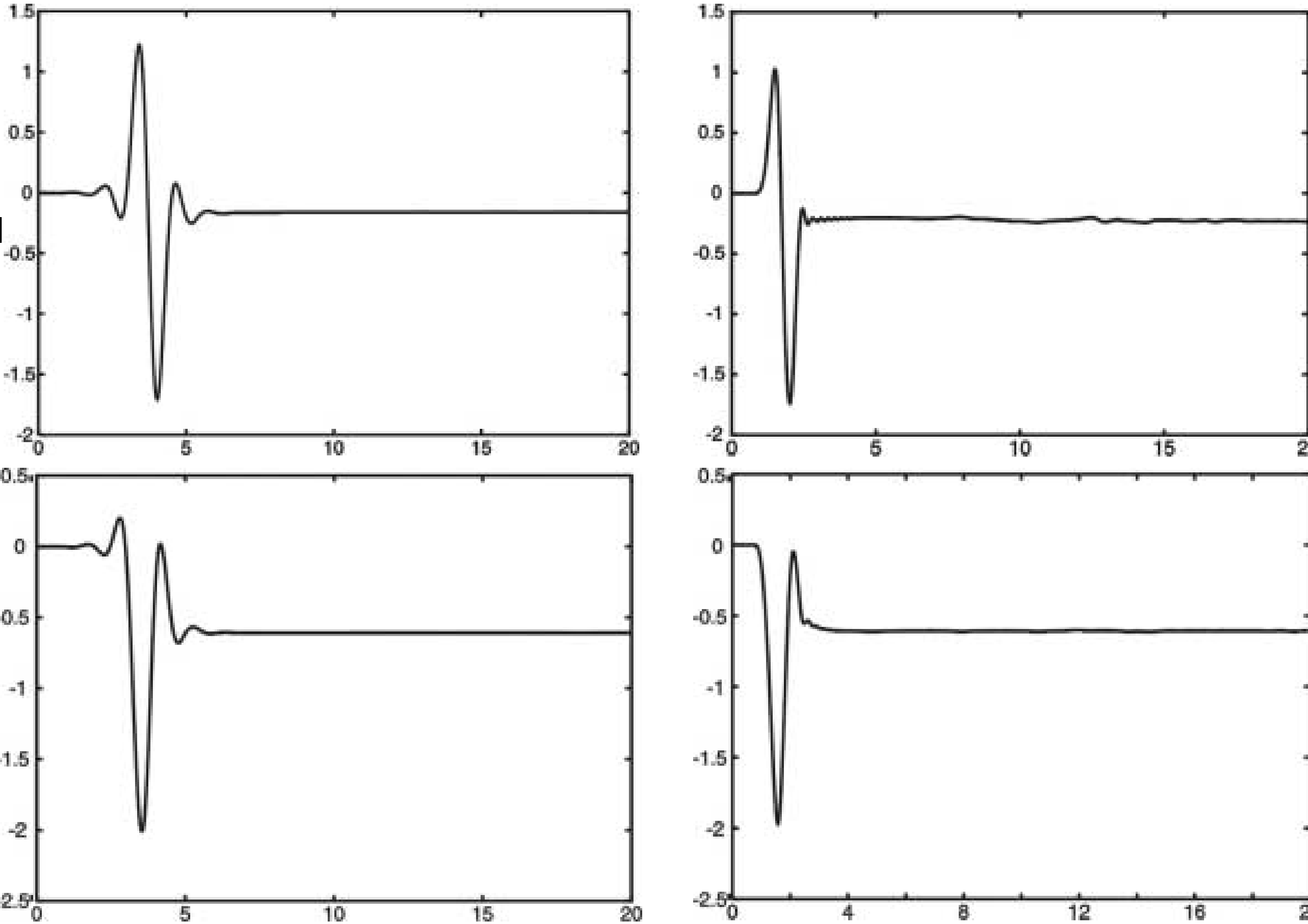


Verification (displacements)



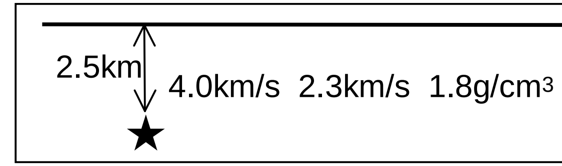
Tangential

Vertical



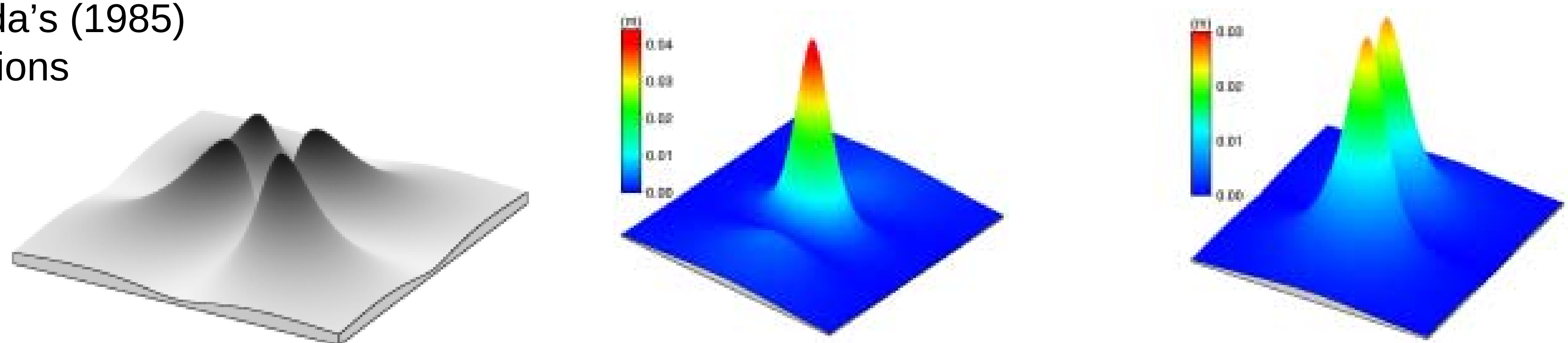
Fk (Zhu & Rivera, 2002) FEM

Verification (static displacements)



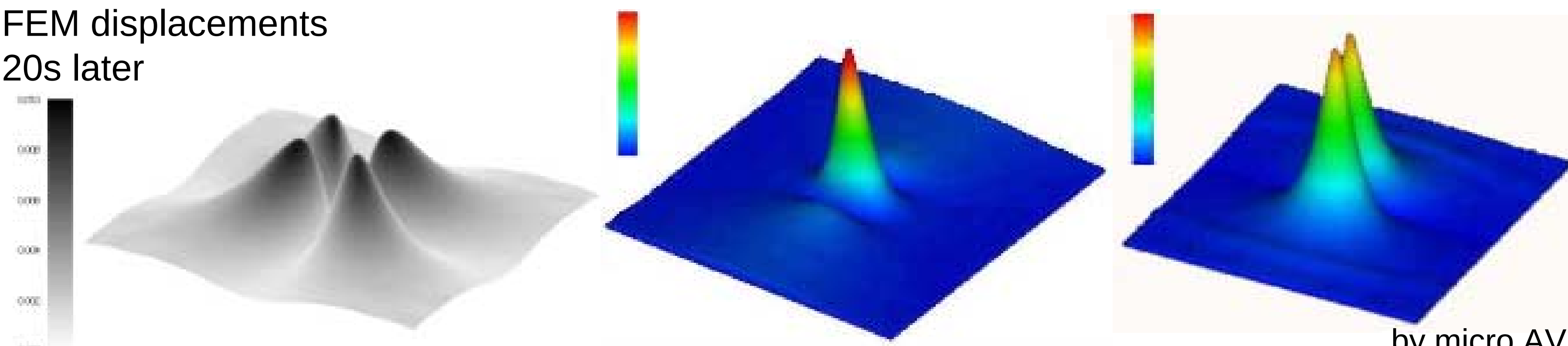
Strike Slip | 45deg Dip Slip | Dip Slip

Okada's (1985) solutions



by GMT

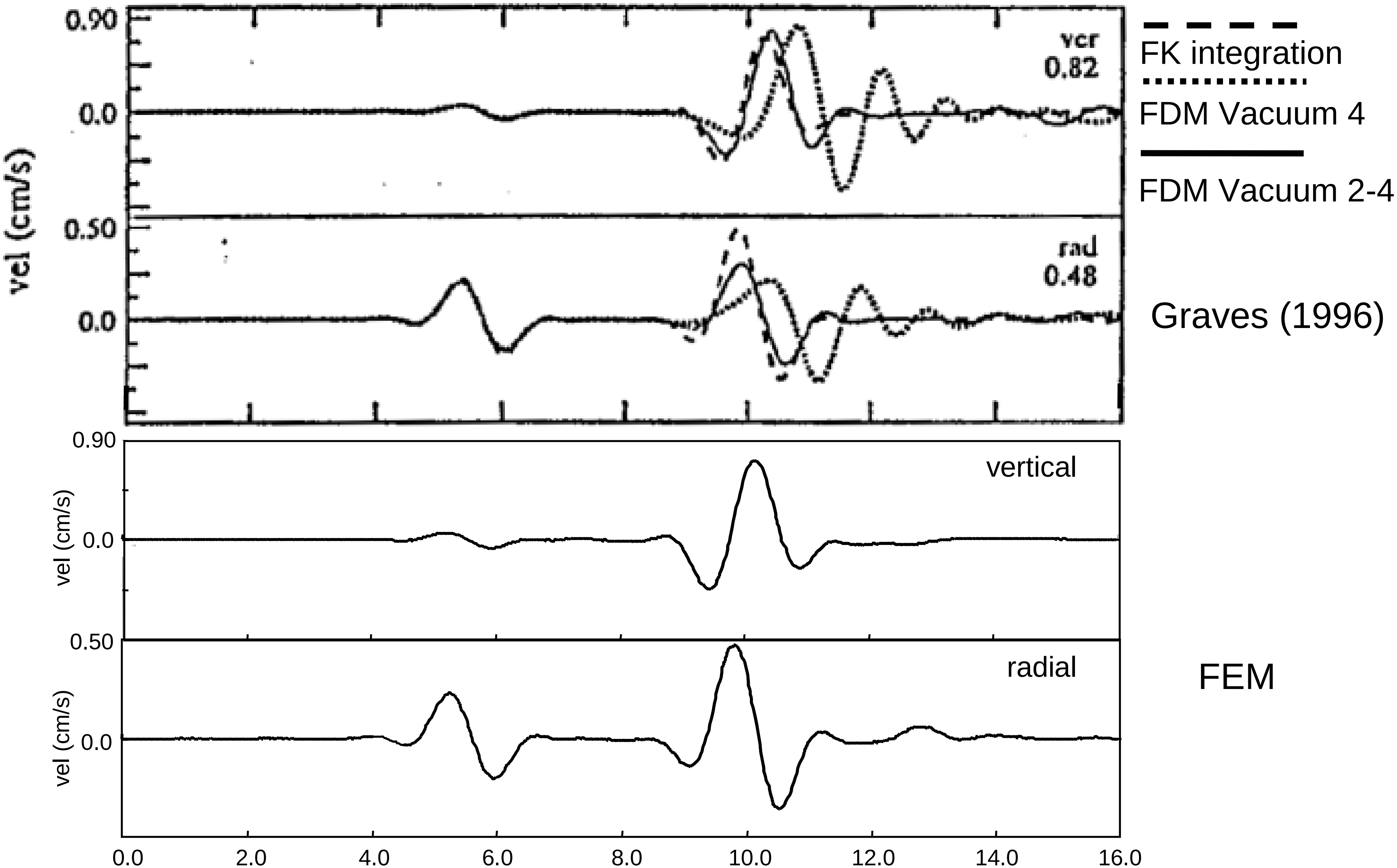
FEM displacements 20s later



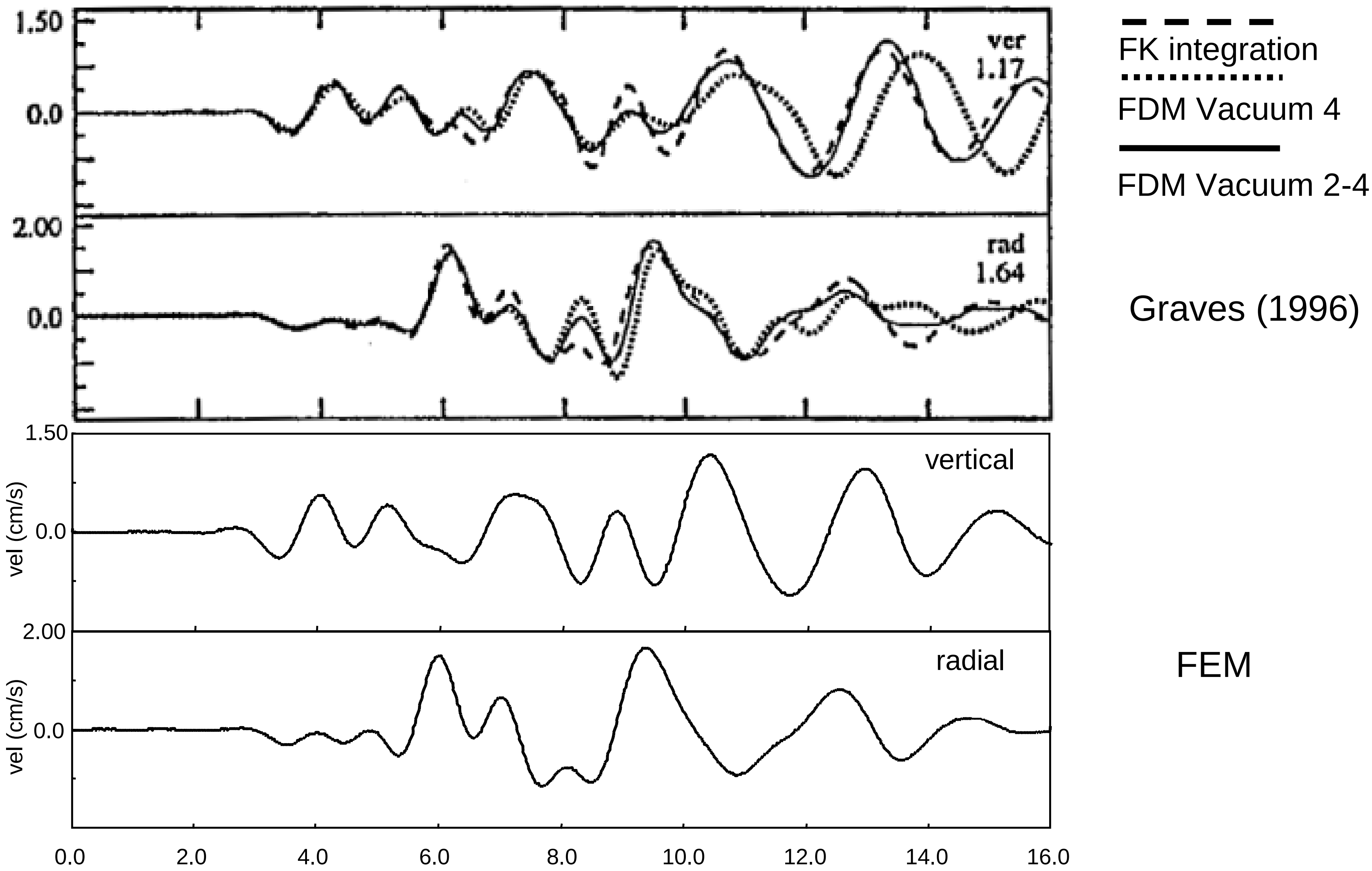
by micro AVS

Absolute Vertical Displacements

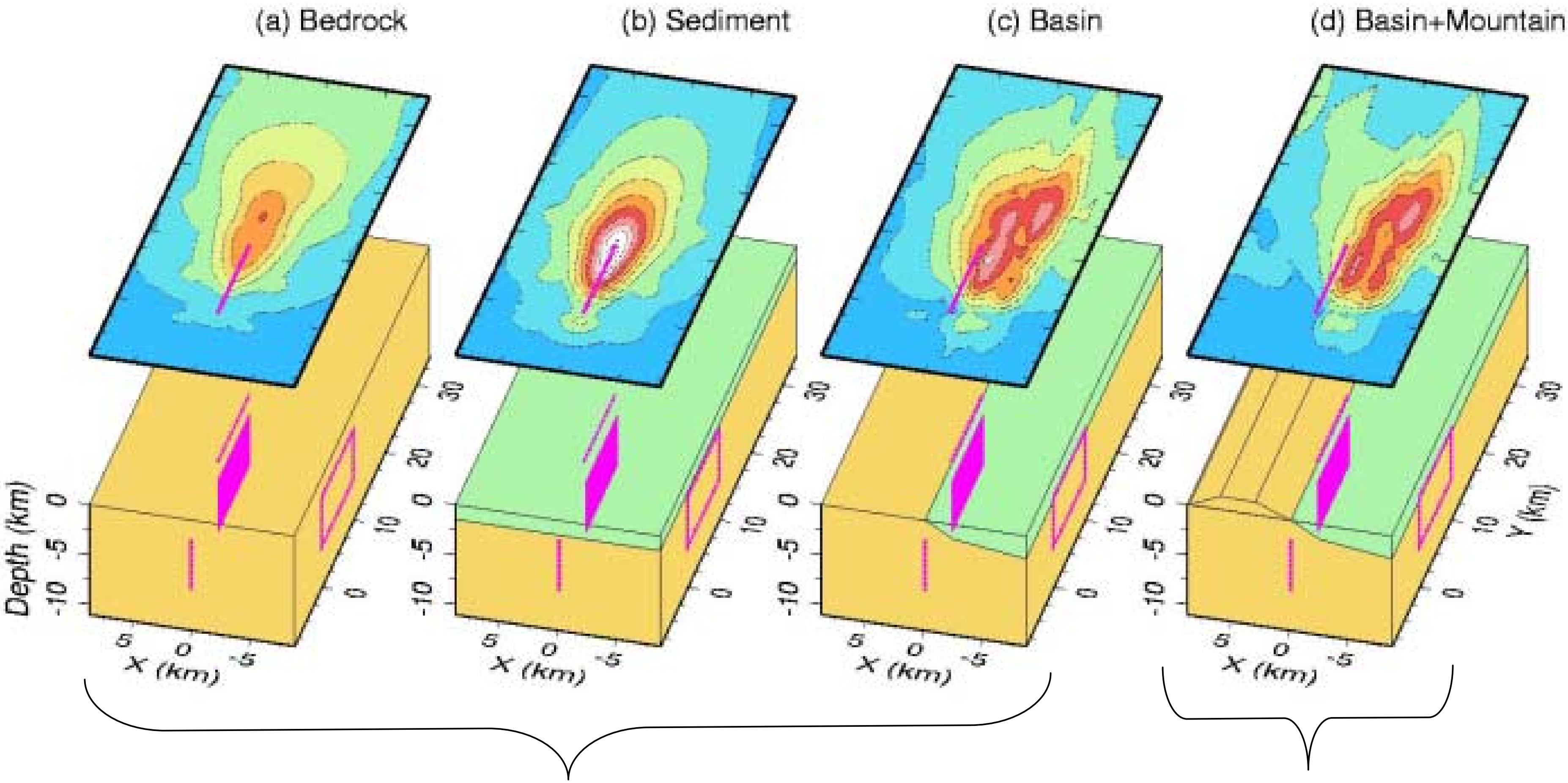
Free Surface (1)



Free Surface (2)



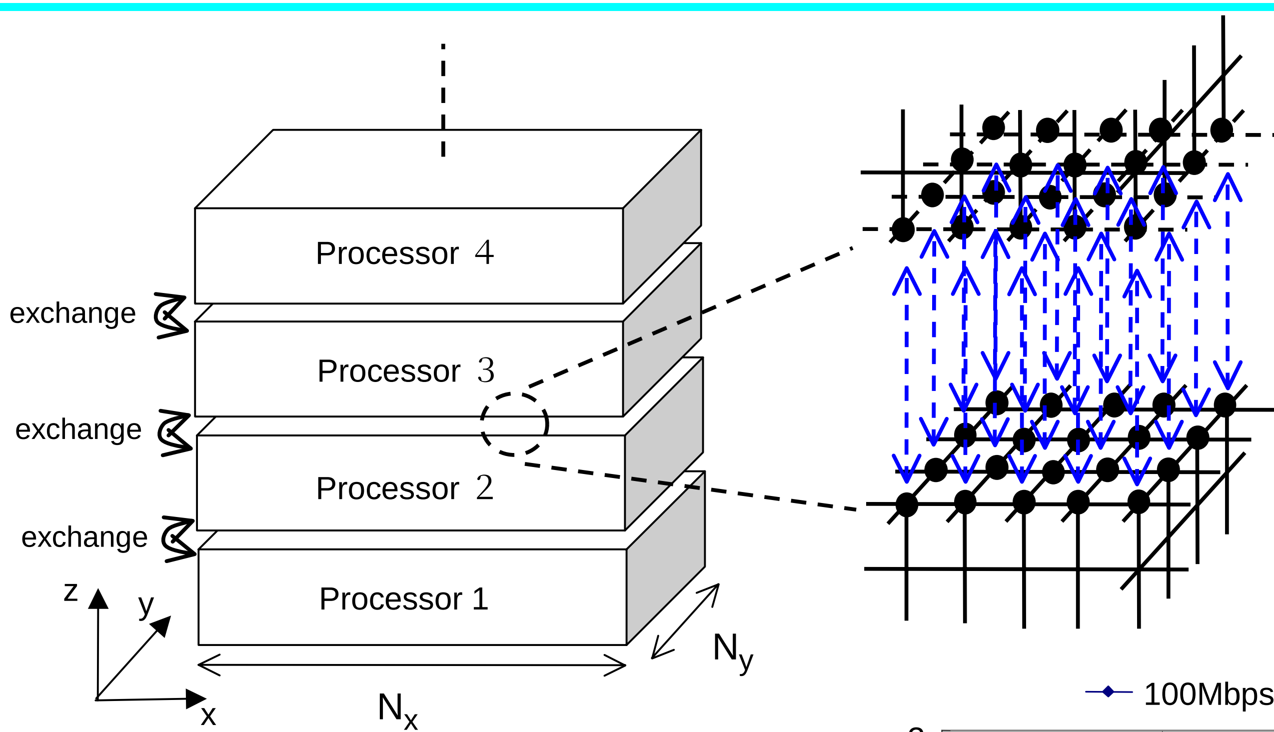
Effect of Surface Topography



FDM (Furumura & Koketsu, 1998)

voxel FEM

Parallel Computing



Data exchange is required only for edge nodes facing each other.