Role of Wavelets in the Physical and Statistical Modelling and Visualization of Complex Geological Processes

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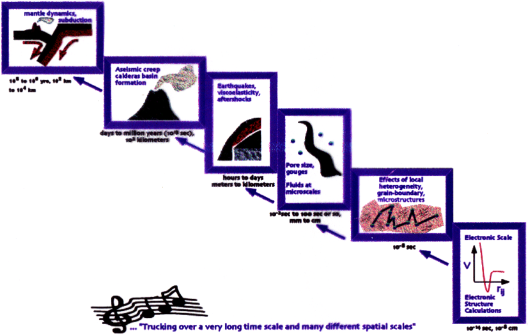
OUTLINE

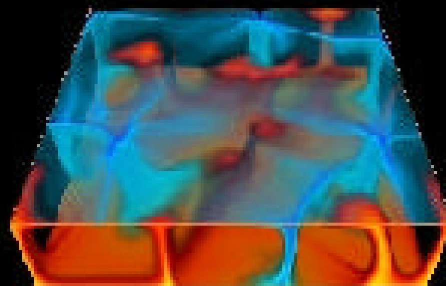
* Introduction :
  multiscale nature of nonlinear problems and the need for using wavelets
* Wavelets and their basic properties
* Use of wavelets in modeling and simulation
* Resolution of scales and Thresholding
* Decomposition into coherent and incoherent parts
* Statistical Analysis
* Visualization
* Conclusions and Perspectives

Cross-scale Processes in Geophysics

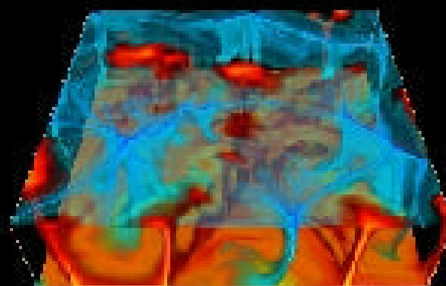


High Rayleigh Number

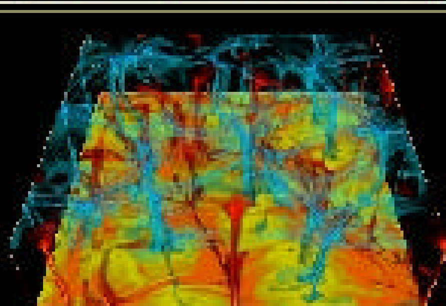
Ra = 10^6



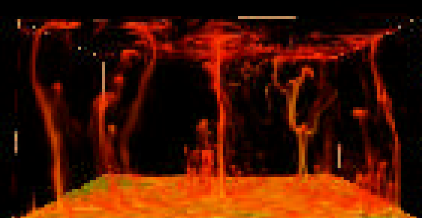
Ra = 10^7



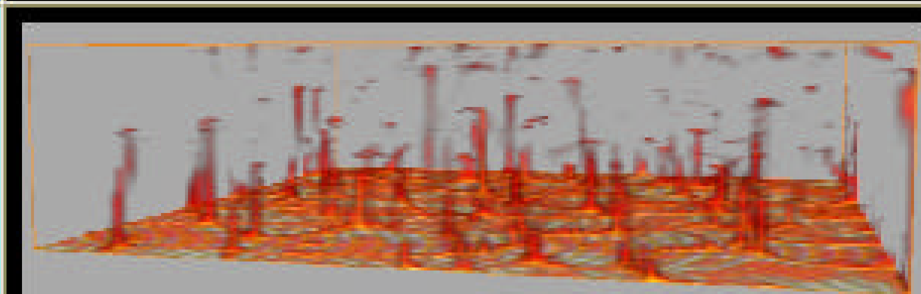
Ra = 10^8



Ra = 10^9

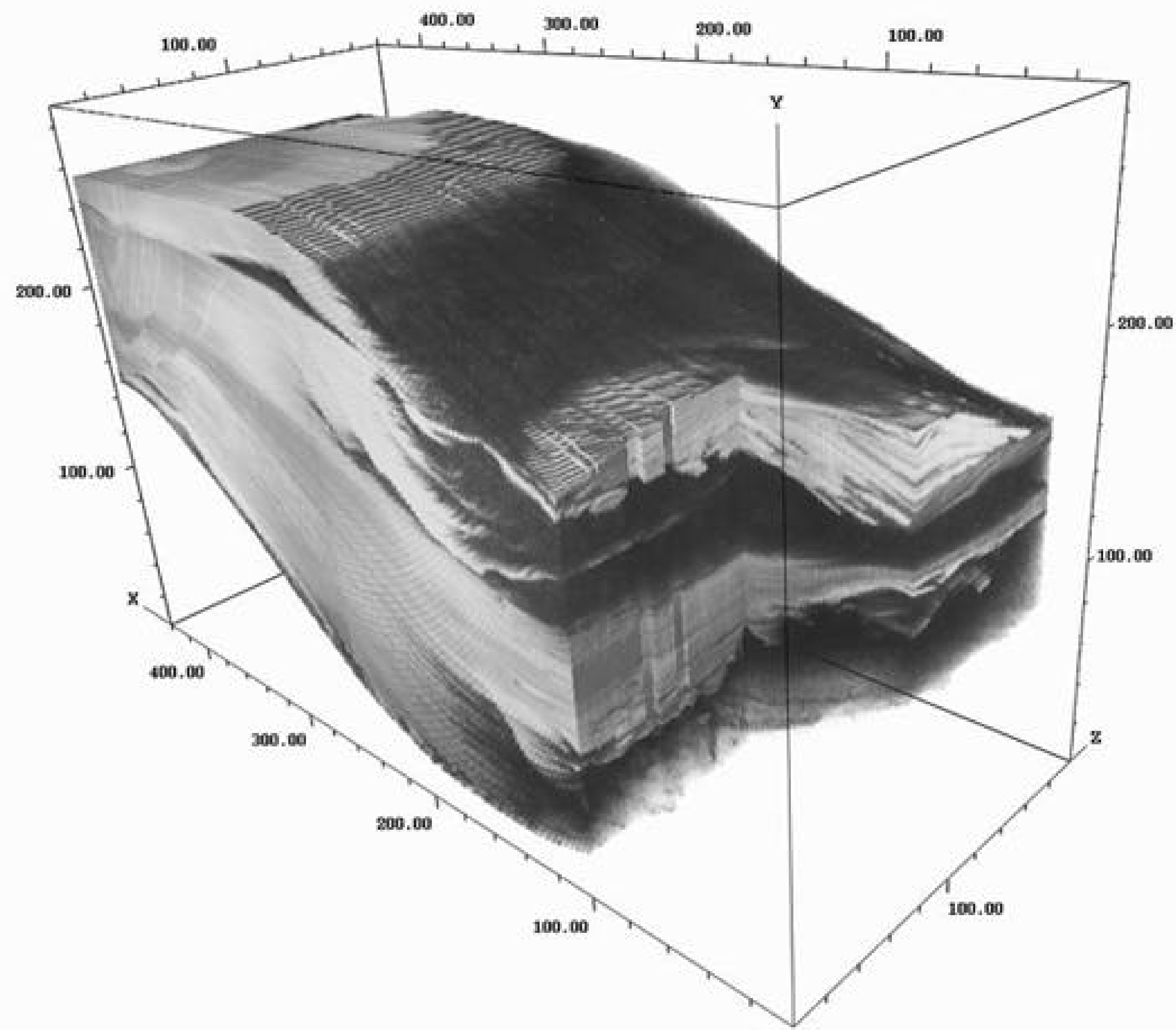


Ra = 10^{10}



Fabien Dubuffet, March 2002

High Rayleigh number 3-D convection for the Earth's mantle from Rayleigh number 10^6 all the way to 10^{10} with the highest resolution of 601x601x601 grid points.

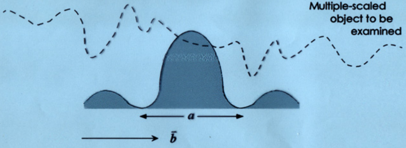


400.00
300.00
200.00
100.00
100.00
Y
200.00
100.00
X
400.00
300.00
200.00
100.00
200.00
100.00
Z
100.00

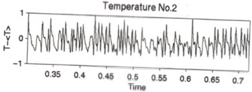
Mother wavelet $\psi\left(\frac{\vec{x}-\vec{b}}{a}\right)$

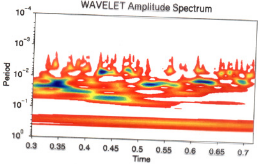
$\vec{x}, \vec{b}$
(multi-dimens.)

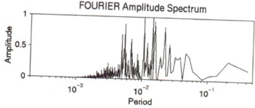
$\vec{b}$ is position vector, a is scale

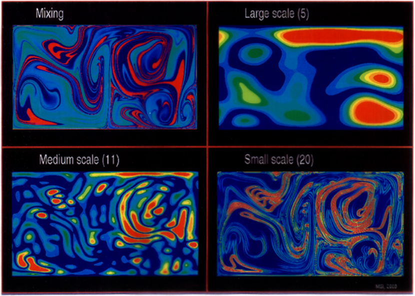


Wavelet transform picks up only the spatial distribution as a function of the moving $\vec{b}$ with length scales of $0(a)$.

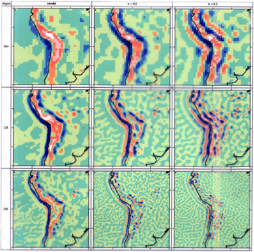

Temperature No.2
T−<T>
1
0
−1
0.35
0.4
0.45
0.5
0.55
0.6
0.65
0.7
Time


WAVELET Amplitude Spectrum
Period
10^{-4}
10^{-3}
10^{-2}
10^{-1}
10^{0}
0.3
0.35
0.4
0.45
0.5
0.55
0.6
0.65
0.7
Time


FOURIER Amplitude Spectrum
Amplitude
1
0.5
0
10^{-3}
10^{-2}
10^{-1}
Period


Mixing
Large scale (5)
Medium scale (11)
Small scale (20)
MSI, 2000

Zoom-in for Chile



$l = 64$

$l = 128$

$l = 256$

wavelet $w = 0.6$ $w = 0.3$

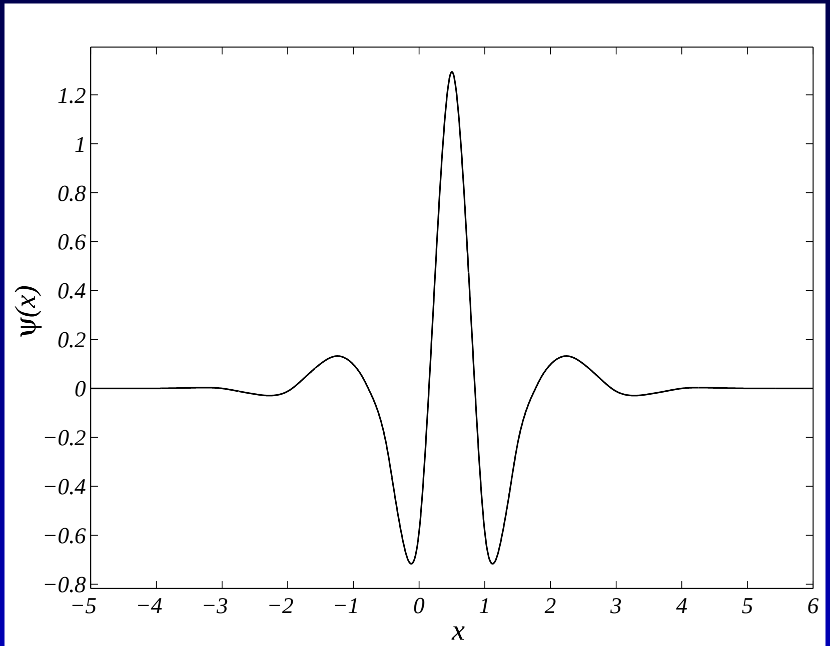
nonlinear filter

First v.s. Second Generation Wavelets?

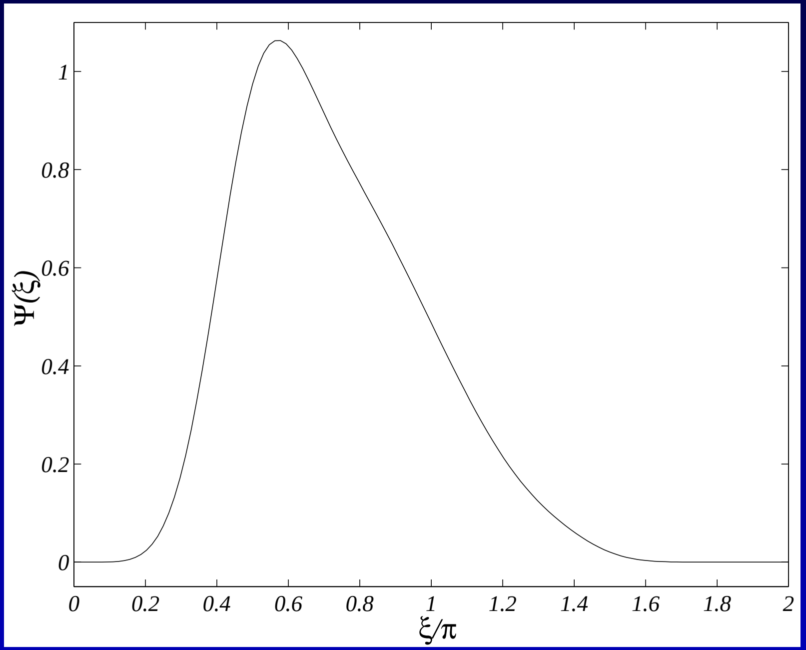
- Weaknesses of first generation wavelets.
  - Periodic or infinite domain.
  - Hard to adapt to complex domains and irregular sampling.
- Second generation wavelets.
  - Constructed in physical space.
  - Can be custom designed for complex multi-dimensional domains and irregular sampling.
  - Factor of two faster than first generation wavelets.
- Additional motivation.
  - Currently have second generation wavelet collocation method to solve PDE's with general boundary conditions.

Second Generation Wavelets

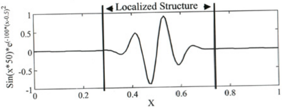
Wavelet



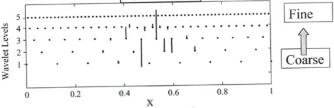
Fourier Transform



Wavelet Thresholding Example



Wavelet Coefficients, $\boxed{c_k^j \geq \varepsilon \equiv 10^{-4}}$,(Compression = 51%)

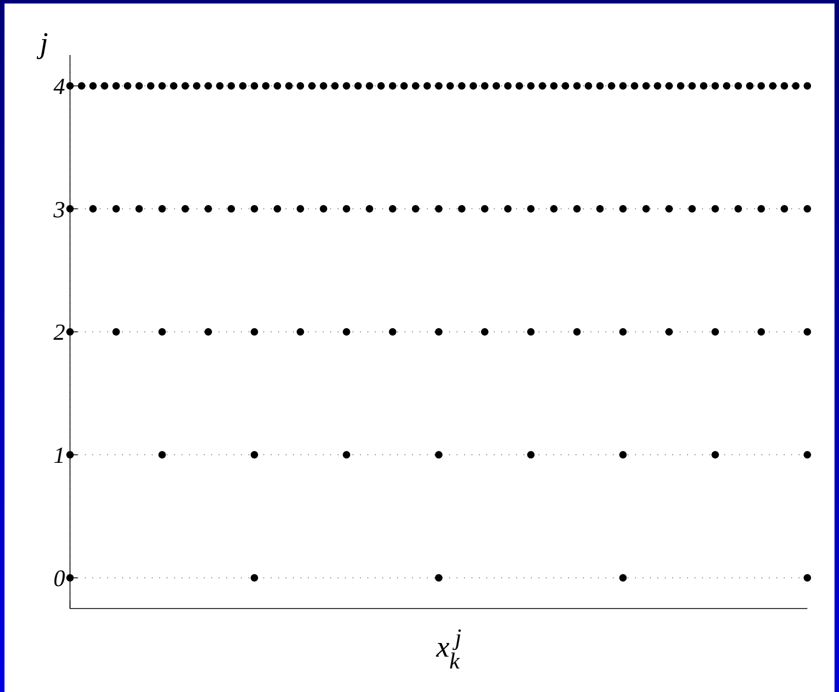


Wavelet Construction

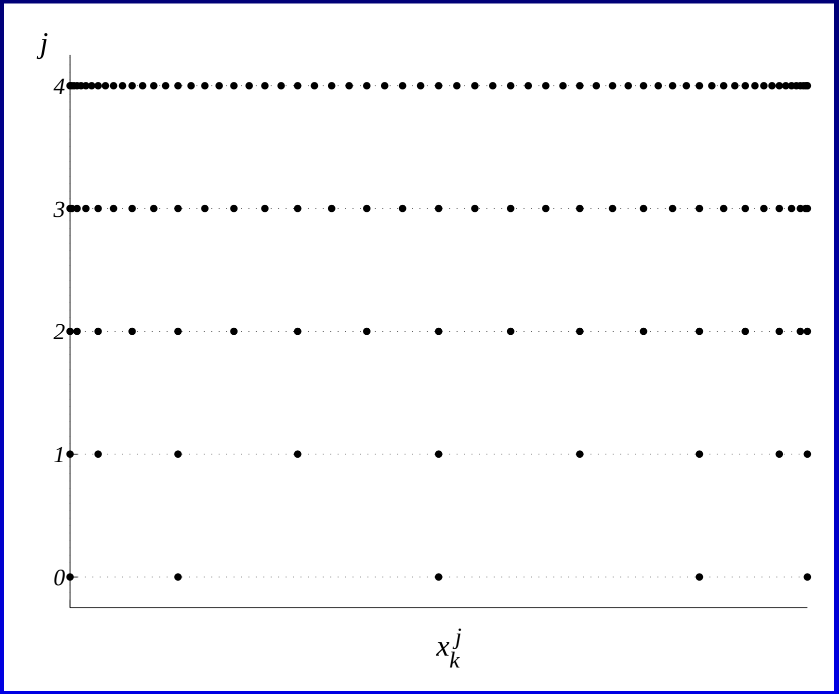
Nested grids:

$$\mathcal{G}^j = \left\{ x_k^j \in \Omega : \; x_k^j = x_{2k}^{j+1}, \; k \in \mathcal{K}^j \right\}, \; j \in \mathcal{J}$$

Uniform Grid



Nonuniform Grid



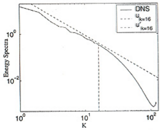
Fourier v.s. Wavelet Filtering

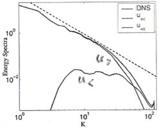
Fourier Cutoff Filter

$$\overline{u}(\mathbf{x}) = 1/\sqrt{2\pi} \sum_{k=0}^{k_{cutof}} \hat{u}(k) e^{ikx}$$

Wavelet Thresholding Filter

$$u_{\geq}(\mathbf{x}) = \sum_{j=0}^{+\infty} \sum_{\substack{k \in \mathcal{K}^j \\ |d_k^j| \geq \epsilon}} d_k^j \psi_k^j(\mathbf{x})$$





Data: DNS, Forced Homogeneous Turbulence $Re_\lambda = 168$

Solving PDEs

$$\mathbf{F}\left(\frac{\partial \mathbf{u}}{\partial t}, \mathbf{u}, \nabla \mathbf{u}, \mathbf{q}, \mathbf{x}, t\right) = 0$$

$$\boldsymbol{\Phi}\left(\mathbf{u}, \nabla \mathbf{u}, \mathbf{q}, \mathbf{x}, t\right) = 0$$

$$u(\mathbf{x}_{\mathbf{k}}^{j}) \implies c_{\mathbf{k}}^{j} \implies \frac{\partial u}{\partial x_i}(\mathbf{x}_{\mathbf{k}}^{j})$$

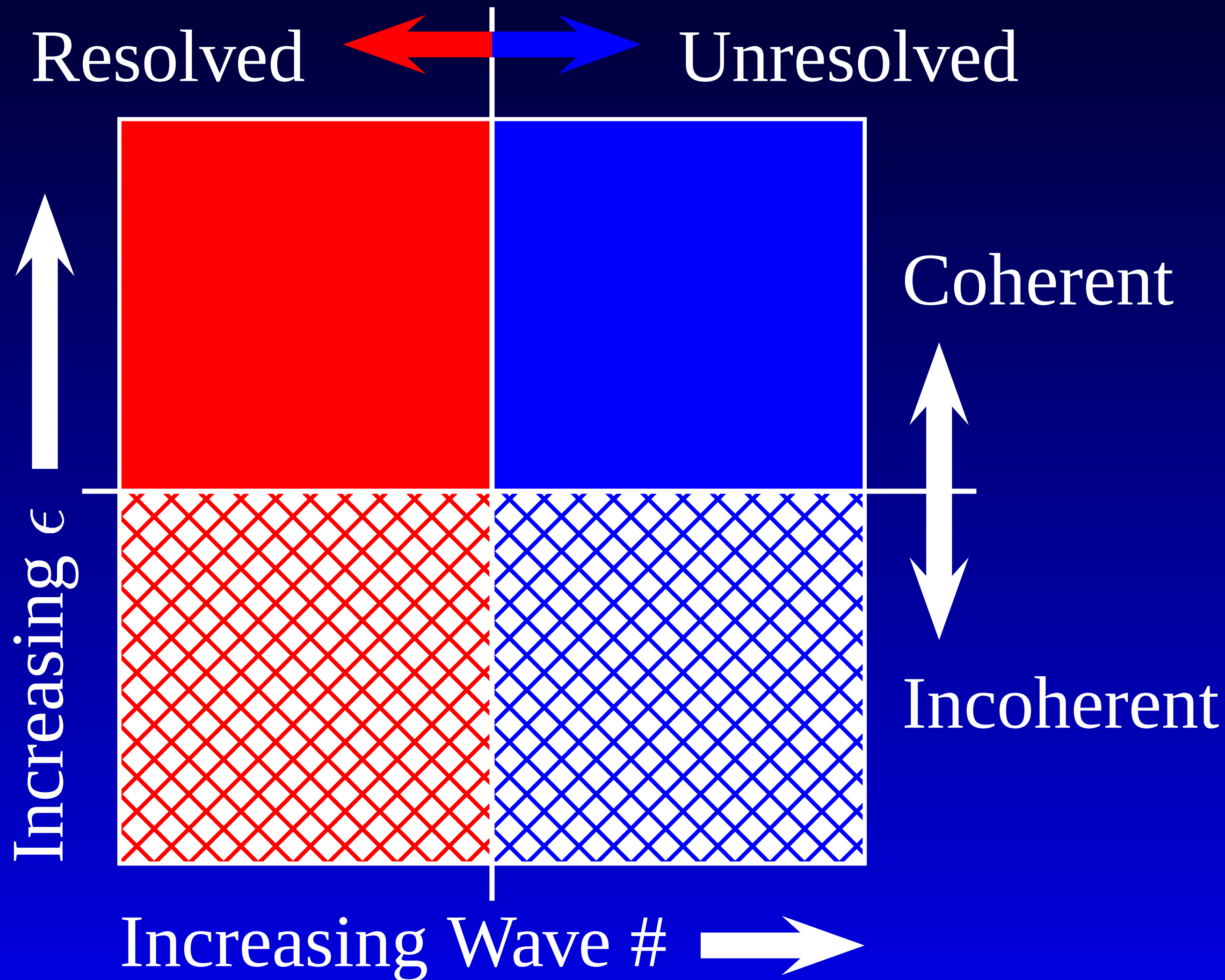
Numerical Algorithm

Elliptic problems:

1. Perform the wavelet transform of $\mathbf{u}_{\mathbf{k}}^{n}$ on $\mathcal{G}_{\geq}^{n}$
2. Update $\mathcal{G}_{\geq}^{n+1}$
3. If $\mathcal{G}_{\geq}^{n+1} = \mathcal{G}_{\geq}^{n}$, go to step 5
4. Interpolate $\mathbf{u}_{\mathbf{k}}^{n}$ to $\mathcal{G}_{\geq}^{n+1}$
5. Solve the system of equations to obtain $\mathbf{u}_{\mathbf{k}}^{n+1}$
6. Stop if $\mathcal{G}_{\geq}^{n+1} = \mathcal{G}_{\geq}^{n}$ and $\|\mathbf{u}_{\mathbf{k}}^{n+1} - \mathbf{u}_{\mathbf{k}}^{n}\| < \delta_{\epsilon}$, otherwise go back to step 1

$\mathcal{G}_{\geq}^{n}$ - computational grid at n-th iteration

Regions of Turbulent Field



$$F\left(\frac{\partial \underset{\sim}{u}}{\partial t}, \underset{\sim}{u}, \quad \underset{\sim}{u}, \ q, \ \underset{\sim}{x}, \ t\right)=0$$

time evolution of vector function $\underset{\sim}{u}$

$$\Phi\,(\underset{\sim}{u}, \quad \underset{\sim}{u}, \ \underset{\sim}{q}, \ \underset{\sim}{x}, \ t)=0$$

(boundary conditions, definition of $\underset{\sim}{q}$,

algebraic differential equations)

$$\underset{\sim}{u} = \underset{\sim}{u}_{>} + \underset{\sim}{u}_{<}$$ ***decomposition***

(coherent, wavelet filtered)	(residual incoherent)

M. Farge (Phys. Fluids, 1999)

Evolution of Coherent Structure $\underset{\sim}{u}>$

$$F\left(\frac{\partial \underset{\sim}{u}>}{\partial t}, \underset{\sim}{u}>, \quad u>, \underset{\sim}{q}, t, \underset{\sim}{x}\right) = f$$

$$\Phi(u>, \quad u, q>, \underset{\sim}{x}, t) = \phi$$

where

$$f = \left[F\left(\frac{\partial \underset{\sim}{u}>}{\partial t}, \underset{\sim}{u}>, \quad u>, \underset{\sim}{q}, \underset{\sim}{x}, t\right) - F\left(\frac{\partial u>}{\partial t}, u>, \quad u>, q>, x, t\right)\right]$$

$$\phi = [\Phi(\underset{\sim}{u}, \quad \underset{\sim}{u}, \underset{\sim}{q}, \underset{\sim}{x}, t) - \Phi(u>, \quad u, q>, \underset{\sim}{x}, t)]$$

is the residual incoherent field forcing that needs to be modeled statistically

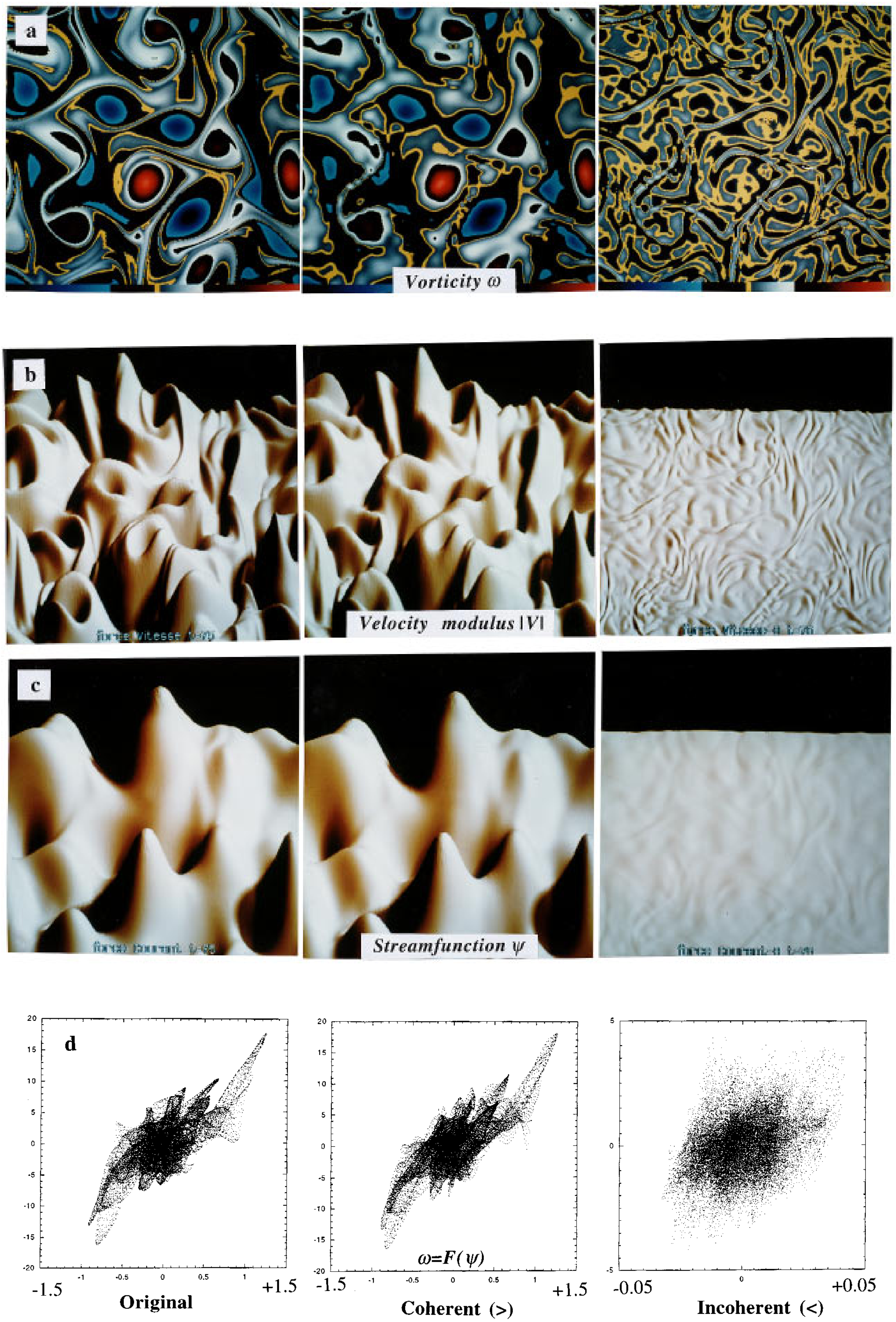
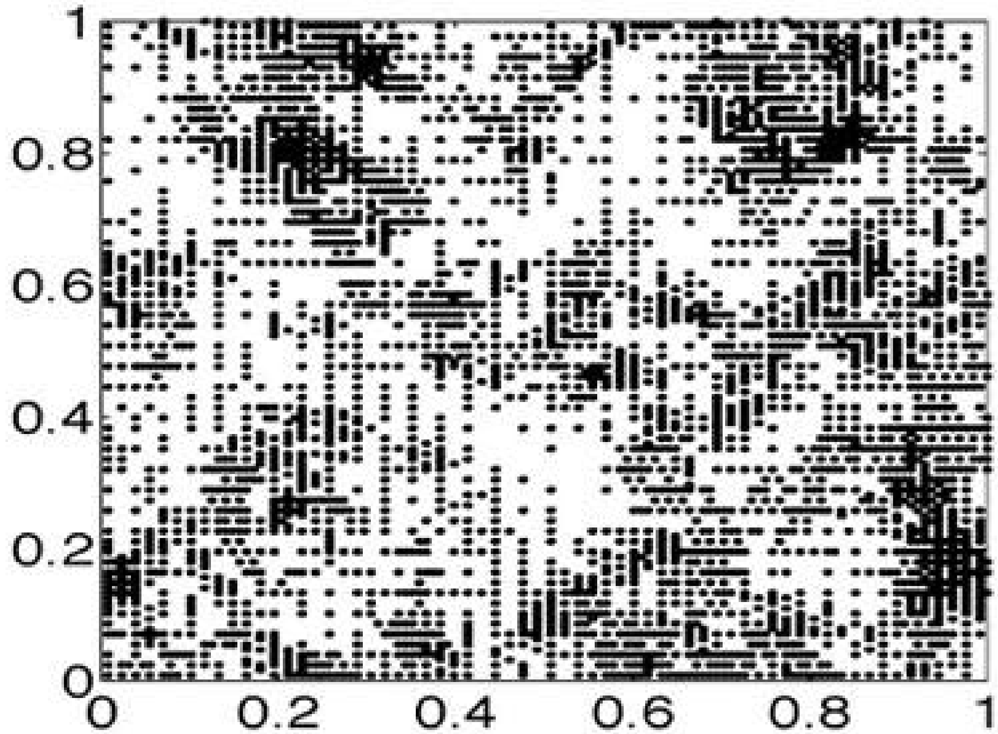
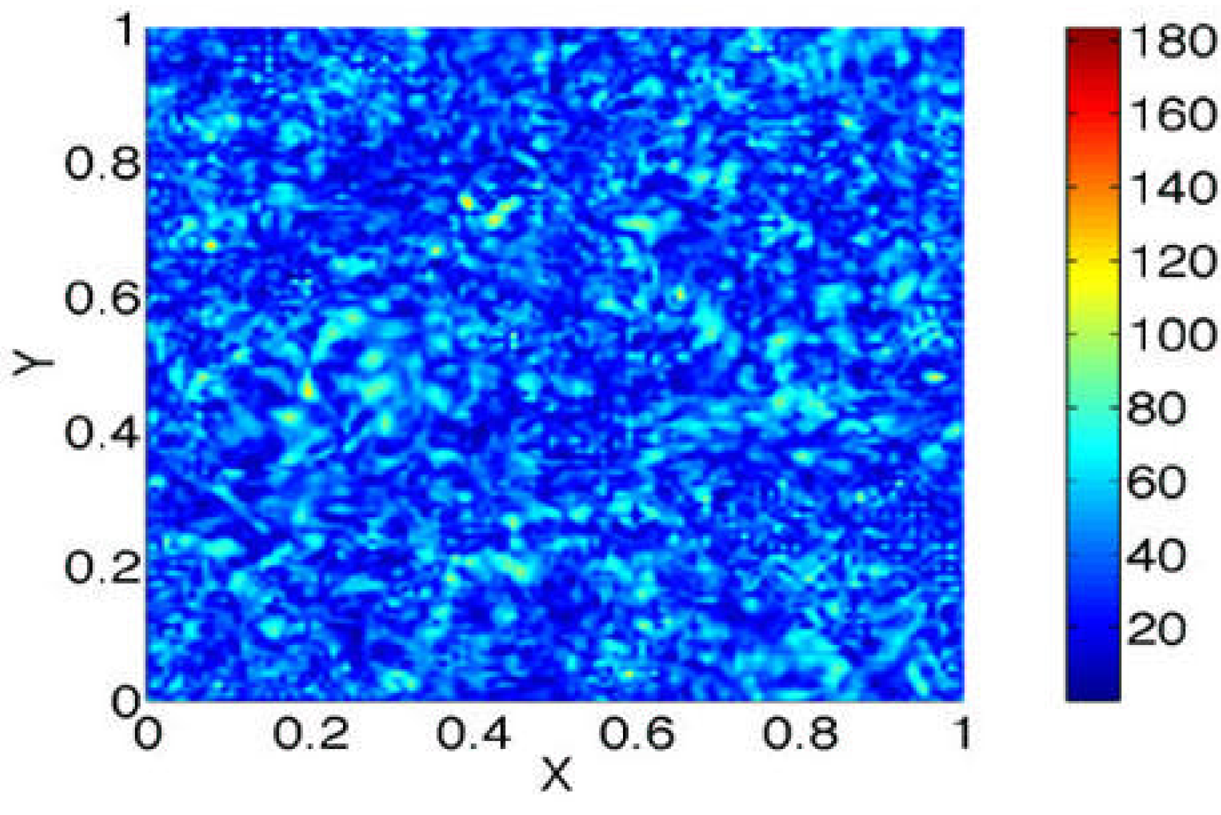


FIG. 2. Wavelet filtering of vorticity. Left: total field. Middle: coherent part. Right: incoherent part. (a) Vorticity ω. (b) Modulus of velocity $|\mathbf{V}|$. (c) Stream function Ψ. (d) Coherence scatter plot ω vs Ψ.

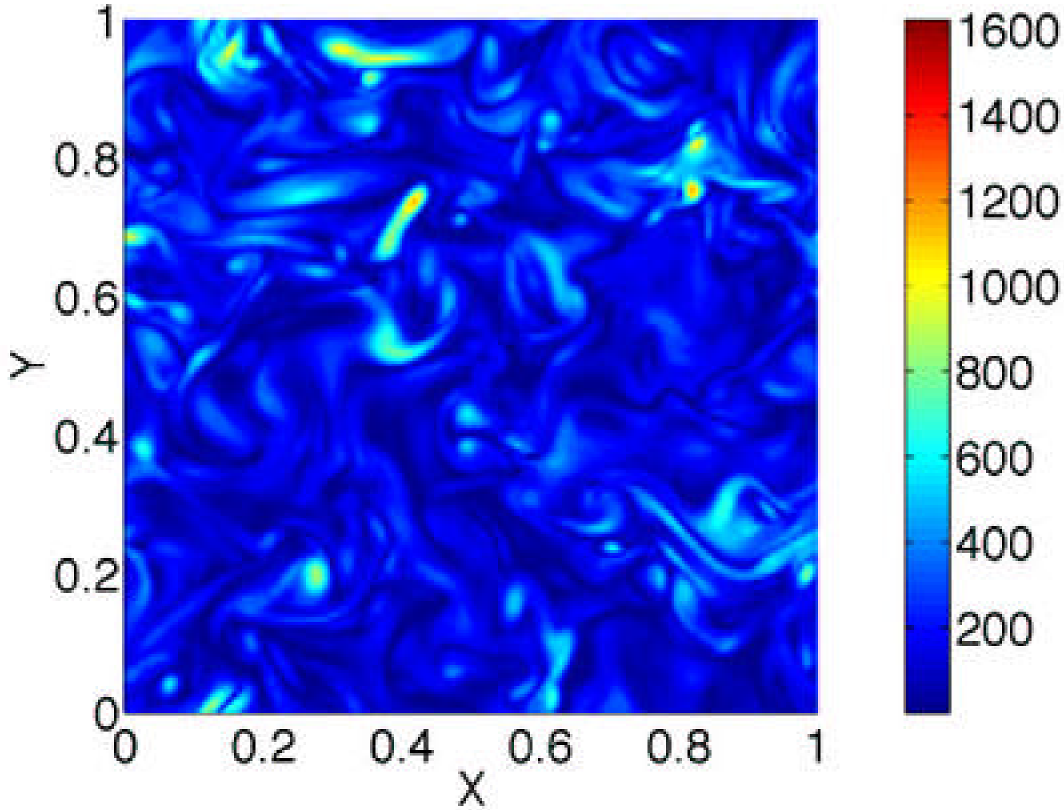
Location of wavelets that form $|\vec{\omega}_>|$



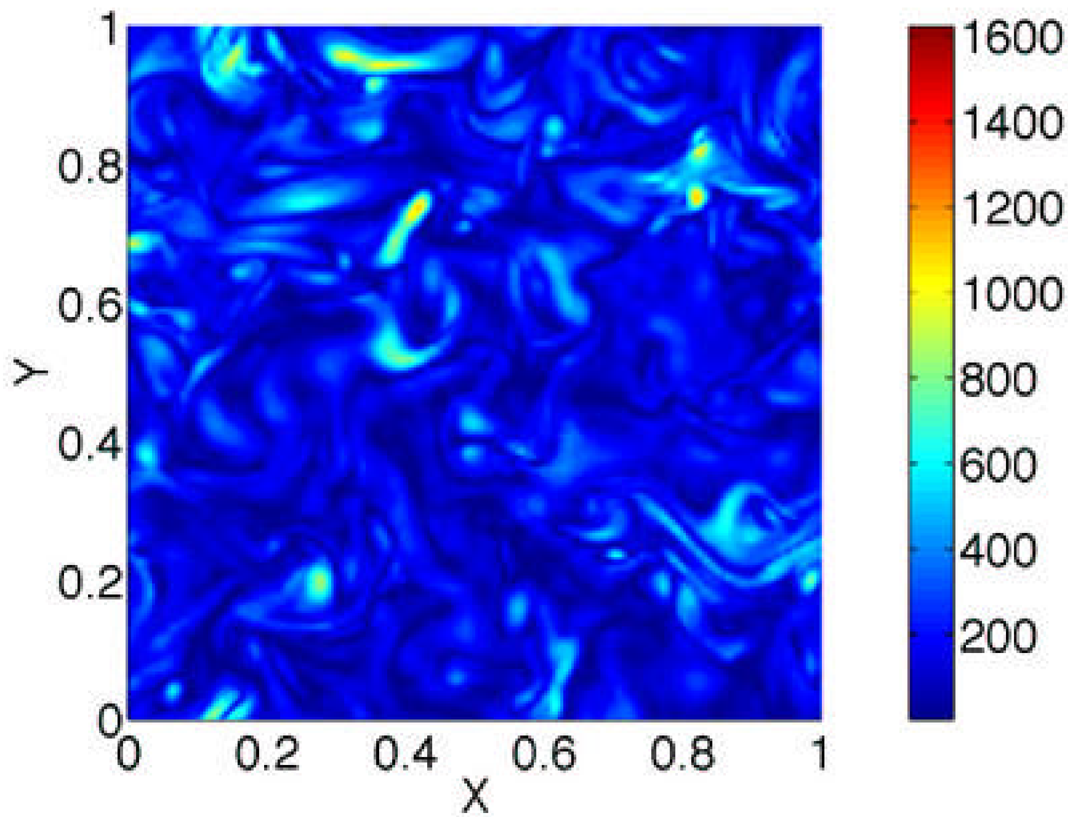
Residual Vorticity $|\vec{\omega}_<|$



vorticity $|\vec{\omega}|$



wavelet-filtered vorticity $|\vec{\omega}_{>}|$



Nonstationary covariance

Classical geostatistics is based on processes which are stationary and isotropic, in the sense that the spatial structure does not change with location.

However, geological processes are rarely stationary and isotropic. Models using wavelets have become very widely used to deal with this lack of stationarity. The motivation for wavelets dates back at least as far as Cohen and Jones (1969),

$$Z(x) = \sum_{\nu=1}^{M} a_\nu \lambda_\nu^{1/2} \psi_\nu(x)$$

where $\{\psi_\nu\}$ are a fixed basis of orthogonal functions, $\{\lambda_\nu\}$ are coefficients to be estimated, and $\{a_\nu\}$ are independent standard normal RVs.

We replace $\{\psi_\nu\}$ by wavelet basis functions, and use a representation of the form (Nychka- and Saltzman, 1998)

$$Z(x) = \sigma(x) \left\{ \rho^{1/2} Z_0(x) + \sum_{\nu=1}^{M} a_\nu \lambda_\nu^{1/2} \psi_\nu(x) \right\}$$

in which $Z_0(x)$ is a stationary isotropic process, ρ is a positive constant and $\sigma(x)$ is a scaling function. The corresponding covariance function is

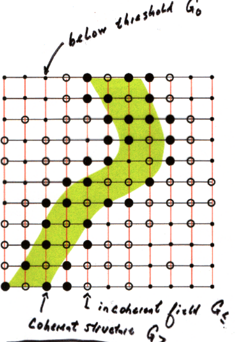
$$C(x_1, x_2) = \sigma(x_1)\sigma(x_2) \left\{ \rho e^{-||x_1 - x_2||/\theta} \right.$$

$$\left. + \sum_{\nu=1}^{M} \lambda_\nu \psi_\nu(x_1) \psi_\nu(x_2) \right\}$$

which permits the standard deviation to vary with location x according to $\sigma(x)$, there is leading term which corresponds to a stationary isotropic model. The remaining terms depend on eigenvalues λ_ν and eigenfunction ψ_ν of the covariance operator an allow various degrees of nonstationarity according to the value of the index M.

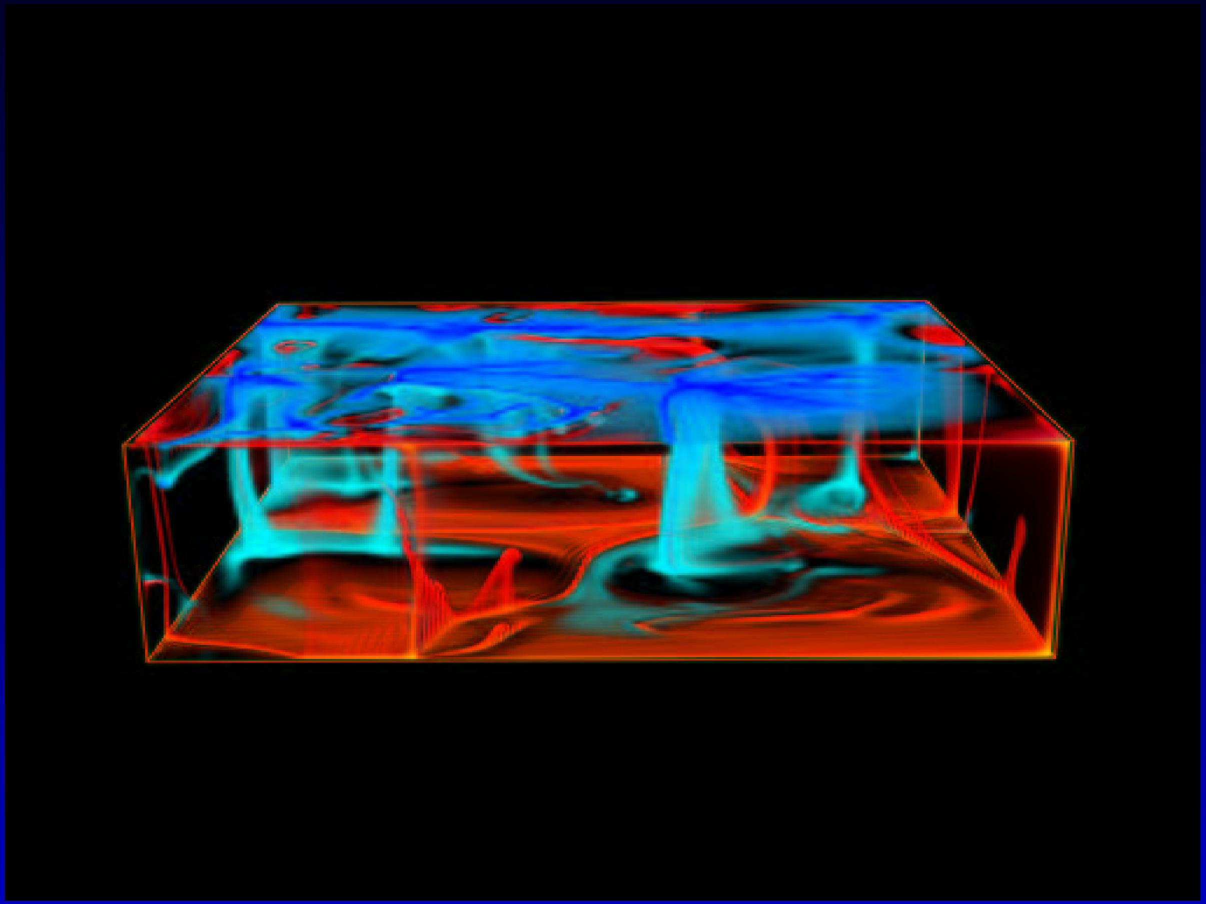
The wavelet representation is motivated by nonstationarity and they also emphasize the computational applicability of the approach in very large systems.

We extend this wavelet approach to non-Gaussian process, and we present a spatial model of heterogeneous geological processes that can help us to improve our understanding of the effect of the incoherent components on the evolution of the coherent structure.

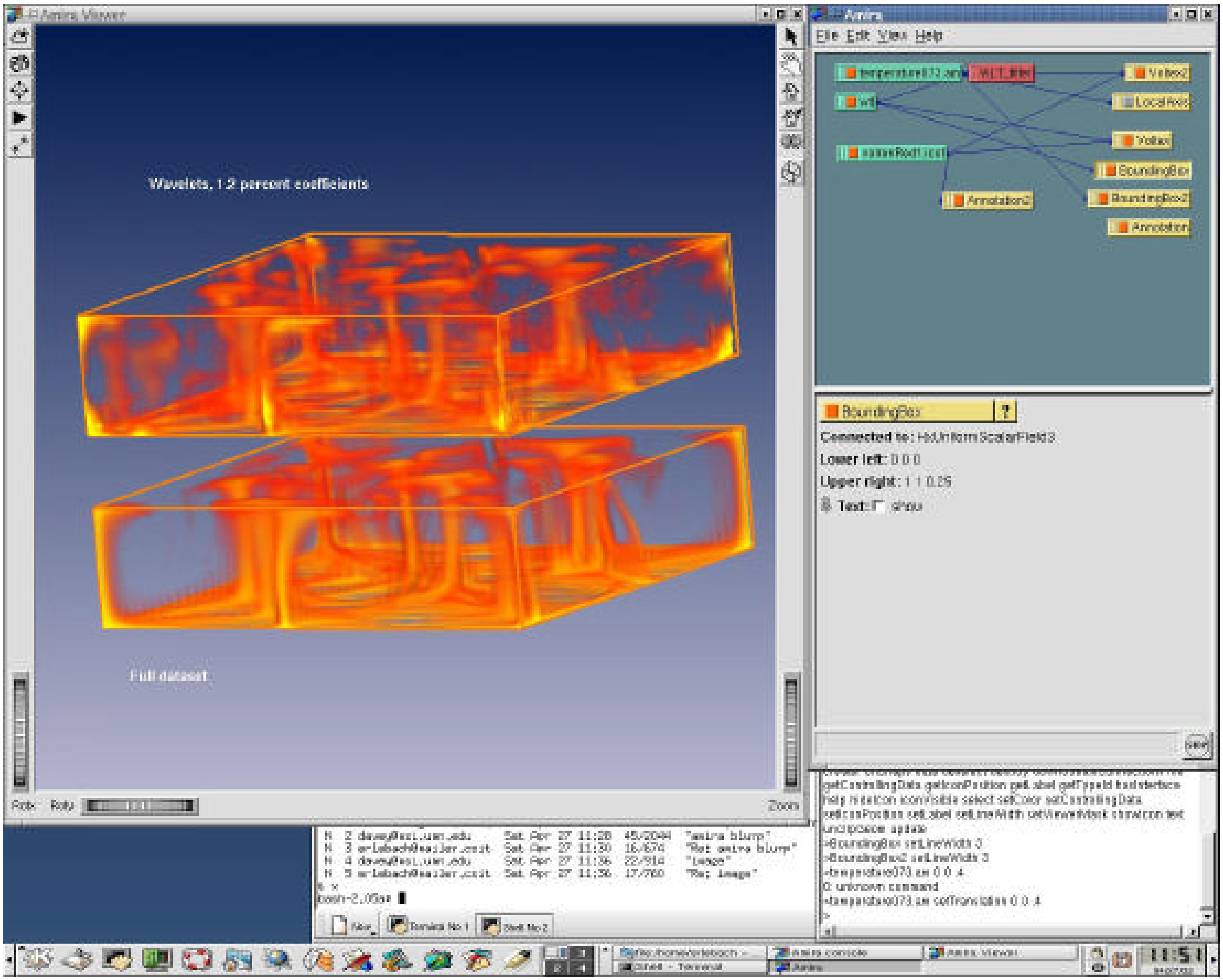


$$\boxed{G = G_0 + G_{>} + G_{\leq}}$$

Motivation: High Rayleigh Number Convection



- 3-D DNS calculation, $Ra = 3 \times 10^8$, $Pr = \infty$
- Computational grid: $513 \times 513 \times 257$
- Results of F. Dubuffet and D.A. Yuen


Amira Viewer
Wavelets, 1.2 percent coefficients
Full dataset
Amira
File Edit View Help
BoundingBox
Connected to: HxUniformScalarField3
Lower left: 0 0 0
Upper right: 1 1 0.25

CONCLUDING REMARKS AND PERSPECTIVES

(1) Wavelets can be used as an efficient adaptive method for numerical simulation of multi-scale phenomena (Vasilyev, Podladchikov, Yuen, Geophys. J. Int., 2001, Vasilyev and Bowman , J. Comput. Phys., 2001) can also use wavelets to tell you where to put the points according to your own scheme (Jameson, L., 1996 in book ed. by G. Erlebacher et al.)

(2.) Wavelet decomposition and de-noising (D. Donoho, 1993) can be used potentially as an effective tool in modelling of complex multiscale physics in geophysics. Deomposition into coherent (organized) and incoherent (random, or Gaussian) fields. We can then MODEL the coherent structures at a much lower computational cost.

CONCLUDING REMARKS AND PERSPECTIVES

(3.) Statistical analysis over irregularly spaced grid which has been decomposed by wavelets to evaluate the effect of the incoherent components.

(4.) Wavelet-based feature extraction, based on thresholding reduces greatly the data storage , between 10 to 100 depending on the degree of non-linearity. This portends well for wireless or mobile computing.

(5.) Wavelets are not everything, beamlets and curvelets to come soon.